# Two simple models for linear set theory

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**Abstract**

In this paper, we give two fairly simple models of set theory with the unrestricted comprehension based on linear logic. The first model is the extension of the idea of Boolean valued models to linear logic. The second model interprets the occurrences of terms and formulas, allowing the interpretations of different occurrences of the same term to be different sets. The soundness of the latter is guaranteed due to the cut-elimination theorem and the absence of contraction.

## 1 Introduction

In this paper, we give two fairly simple models of set theory with the unrestriected comprehension based on linear logic (for the proof-theoretic study, see [3, 4, 5, 6, 7, 8, 9]) The first model is the extension of the idea of Boolean-valued models to linear logic. The second model interprets the occurrences of terms and formulas, allowing the different interpretations for the two occurrences of the same term. We show the completeness for the first model, and only the soundness for the second.

Komori [5] defined a semantics for set theory in affine (BCK) logic in terms of Kripke models, and proved completeness of his system with respect to it. Our first model is in a similar vein to his work. The second model seems to be related to the idea of stratification in Quine's NF [1].

In the following, we first review the standard approach to the semantics of set theory based on a nonstandard logic and extend it to a system SIM of linear set theory. The result is our first model, and we show the completeness of SIM with respect to it. Secondly, we introduce a sytem SLIM of linear set theory

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with the strict (or linear) comprehension. We then define the interpretation of the ocurences of terms of SLIM in the hereditarily finite extension $\mathrm{V}_\omega(A)$ of the classical set-theoretical universe $A$ and show its soundness.

## 2 Phase-valued models of linear set theory

The standard approach to the semantics of set theory based on a nonstandard logic is the extension of the idea of Boolean-valued models for classical set theory. Given a nonstandard logic $L$, let $M$ be a member of the class of algebra to which the Lindenbaum algebra of $L$ belongs. Then an $M$-valued model $V^M$ of set theory based on $L$ is defined as a pair $(V, \in)$ where $V$ is a class and $\in$ is a binary operation from $V \times V$ to $M$. For example, we can use a Heyting algebra as $M$ for intuitionistic logic, and an ortholattice for quantum logic. The typical way to construct such a model is by transfinite induction as follows:

1. $V_o = \emptyset$
2. $V_{\alpha+1} = \{f : f \text{ is a partial function from } V_\alpha \text{ to } M\}$
3. $V_\lambda = \bigcup_{\gamma \in \lambda} V_\gamma$ where $\lambda$ is a limit ordinal,
4. $V = \bigcup_{\gamma \in Ord} V_\gamma$

This method of constructing an $M$-valued model is also applicable to linear set theory. We can, for example, use a phase space as $M$ for classical linear logic or a quantale for intuitionistic linear logic [2]. The model thus constructed indeed verifies those sentences of linear set theory which we can regard as the linear version of the axioms of Zermelo-Fraenkel set theory [6]. However, the model does not verify the unrestricted comprehension due to its cumulative nature.

For the unrestricted comprehension, we need a universe $U$ which satifies $U \cong [U \to M]$, where $[U \to M]$ is a suitable function space closed under the operations of linear logic. This equation looks very similar to a domain equation, and one may think that we can construct such $U$ by a method similar to Scott's construction of $D^\infty$. However, an element of a function space is required to be a monotone function in Scott's method, and this conflicts with closure condition under linear negation. It may be that a simple modification of the method suffices, but we postpone this line of investigation to another occasion.

For now, we only give a specification of the required phase-valued models in a way similar to the definition of $\lambda$-models, and show the completeness of a linear set theory with respect to it. For the sake of exposition, we only consider the very simple system of linear set theory SIM.

**Definition 1** *The terms and formulas of SIM are defined by simultaneous induction as follows:*

1. *Variables $x$, $y$, $z$, ... are terms;*

2. *If $s$ and $t$ are terms, then $s \in t$ and $s \notin t$ are formulas which are atomic;*

3. *If $A$ is a formula, then $\{x : A\}$ is a term;*

4. *If $A$ and $B$ are formulas, then $A \otimes B$ and $A \parr B$ are formulas.*

*We write $Exp$ for the set of all terms and formulas, and $Var$ for the set of all variables.*

The duals $A^{\perp}$ are defined in the standard manner. The axioms and the rules of inference of SIM are given as follows.

**Axioms:**

$$\vdash s \in t, s \notin t$$

**The rules of inference:**

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}$$

$$\frac{\vdash A[s/x], \Gamma}{\vdash s \in \{x : A\}, \Gamma} \qquad \frac{\vdash A[s/x], \Gamma}{\vdash s \notin \{x : A^{\perp}\}, \Gamma}$$

$$\frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

**Proposition 2** *SIM allows cut-elimination.*

**Proof**
By induction on the size of proofs. ∎

We now define a class of phase-valued models for SIM.

**Definition 3** *A phase space $\mathcal{P}$ is the quadruple $(P, \cdot, 1, \perp)$ where*

1. *$(P, \cdot, 1)$ is a commutative monoid,*

2. *$\perp \subset P$.*

*We often write $pq$ for $p \cdot q$ with $p$, $q \in P$.*

**Definition 4** *Let $(P, \cdot, 1, \perp)$ be a phase space. We define the operations $\otimes$, $\parr$ and $(\ )^{\perp}$ on the powerset of $P$ as follows:*

1. *$A \otimes B =_{def} \{pq : p \in A \text{ and } q \in B\}^{\perp\perp}$*

2. *$A^{\perp} =_{def} \{p : \text{For all } q \in A (pq \in \perp)\}$*

3. $A \multimap B =_{def} \{p : For\ all\ q \in A(pq \in B)\}$

4. $A \parr B =_{def} (A^{\perp} \otimes B^{\perp})^{\perp}$

*A subset $A$ of $P$ is called a fact if $A^{\perp\perp} = A$. We write $FP$ for the collection of all facts in $\mathcal{P}$.*

**Definition 5** *A phase-valued model of SIM is the quadruple $(V, \in, \mathcal{P}, [\![\ ]\!])$ such that*

1. *$V$ is a set.*

2. *$\mathcal{P} = (P, \cdot, 1, \perp)$ is a phase space.*

3. *$\in$ is a function from $V \times V$ to $P$.*

4. *Let $\Upsilon$ be the set of all functions from $Var$ to $V$. The members of $\Upsilon$ are called assignments. Then $[\![\ ]\!]$ is a function from $Exp \times \Upsilon$ to $F(\mathcal{P}) \cup V$ satisfying*

   (a) $[\![s]\!]_\eta \in V$ *for every term* $s$

   (b) $[\![A]\!]_\eta \in P$ *for every formula* $A$

   (c) $[\![x]\!]_\eta = \eta(x)$

   (d) $[\![s \in t]\!]_\eta = [\![s]\!]_\eta \in [\![t]\!]_\eta$

   (e) $a \in [\![\{x : A\}]\!]_\eta = [\![A]\!]_{\eta[x \mapsto a]}$ *for every* $a \in V$

   (f) $[\![A \otimes B]\!]_\eta = [\![A]\!]_\eta \otimes [\![B]\!]_\eta$

   (g) $[\![A^{\perp}]\!]_\eta = ([\![A]\!]_\eta)^{\perp}$

   (h) $[\![e_1]\!]_{\eta[x \mapsto [\![e_2]\!]_\eta]} = [\![e_1[e_2/x]]\!]_\eta$ *for every* $e_1$, $e_2 \in Exp$

**Definition 6** *Let $\mathcal{V} = (V, \in, \mathcal{P}, [\![\ ]\!])$ be a phase-valued model of SIM. A formula $A$ of SIM is valid in $\mathcal{V}$ if $1 \in [\![A]\!]_\eta$ for every assignment $\eta$ in $\mathcal{P}$. If $A$ is valid in every phase-valued model of SIM, we call it valid.*

**Proposition 7** *Let $\Gamma^\star$ be the formula obtained by combinig all the occurrences of the formulas in $\Gamma$ by the connective $\parr$. Suppose that $\vdash \Gamma$ is provable in SIM. Then, $\Gamma^\star$ is valid.*

**Proof**
Classical propositional linear logic is sound with respect to the phase space semantics. So, it suffices to verify $[\![s]\!]_\eta \in [\![\{x : A\}]\!]_\eta = [\![A[s/x]]\!]_\eta$. But

$$[\![s]\!]_\eta \in [\![\{x : A\}]\!]_\eta = [\![A]\!]_{\eta[x \mapsto [\![s]\!]_\eta]} = [\![A[s/x]]\!]_\eta$$

∎

We also show the completeness of SIM with respect to the above class of phase-valued models. For this purpose, we define the term model of SIM.

**Definition 8** *The phase space $\mathcal{M}$ generated by SIM is the quadruple $(M, \cdot_M, 1_M, \bot M)$ where*

1. *$M$ is the set of all multisets of formulas of SIM.*
2. *$\cdot_M$ is the concatenation of multisets.*
3. *$1_M$ is the empty multiset.*
4. *$\bot_M$ is the set of $\Gamma$ such that $\vdash \Gamma$ is provable in SIM.*

**Definition 9** *Let $A$ be a formula of SIM. The canonical interpretation $S(A)$ of $A$ in $\mathcal{M}$ is defined as $S(A) = \{\Gamma : \vdash A, \Gamma$ is provable in SIM$\}$.*

**Proposition 10** *For every formula $A$ of SIM, $S(A)$ is a fact in $\mathcal{M}$.*

**Proof**
$S(A) \subset A^{\bot\bot}$ always holds. So, it suffices to prove $S(A)^{\bot\bot} \subset S(A)$. We first show $S(A)^{\bot} = S(A^{\bot})$. Let $\Gamma \in S(A)^{\bot}$. Since $\vdash A, A^{\bot}$ is always provable in SIM, we have $A^{\bot} \in S(A)$. Then $\vdash A^{\bot}, \Gamma$ is provable by the definition of $S(A)^{\bot}$. Hence $\Gamma \in S(A^{\bot})$. On the other hand, let $\Gamma \in S(A^{\bot})$. Then $\vdash A^{\bot}, \Gamma$ is provable in SIM. Suppose $\Delta \in S(A)$. Then $\vdash A, \Delta$ is provable in SIM. So, we can derive $\vdash \Gamma, \Delta$ by cut. Therefore, the concatenation of $\Gamma$ and $\Delta$ is in $\bot_M$. Hence $\Gamma \in S(A)^{\bot}$.

Now we show $S(A)^{\bot\bot} = S(A^{\bot})^{\bot} \subset S(A)$. Let $\Gamma \in S(A^{\bot})^{\bot}$. Since $\vdash A, A^{\bot}$ is provable, $A \in S(A^{\bot})$. Hence, $\vdash A, \Gamma$ is provable in SIM. So, $\Gamma \in S(A)$. ∎

**Definition 11** *The term model $\mathcal{T}$ of SIM is the quadruple $(T, \in_T, \mathcal{M}, [\![\ ]\!]^{\mathcal{T}})$ such that*

1. *$T$ is the set of all closed terms of SIM.*
2. *$s \in_{\mathcal{T}} t = S(s \in t)$ for every closed terms $s$ and $t$ of SIM.*
3. *Let $\eta$ be an assignment from $Var$ to $T$. Then,*
   (a) *$[\![A]\!]^{\mathcal{T}}_{\eta} = S(A_{\eta})$ for every formula $A$*
   (b) *$[\![A]\!]^{\mathcal{T}}_{\eta} = s_{\eta}$ for every term $s$,*

   *where $A_{\eta}$ and $s_{\eta}$ are obtained from $A$ and $s$, respectively, by the substitution of closed terms for variable according to $\eta$.*

**Lemma 12** *The term model $\mathcal{T}$ is a phase-valued model of SIM.*

**Proof**
The propositional part is verified as in Girard's original paper. The conditions (a)-(c) are satisfied by the definition. Hence, it suffices to verify the conditions (d), (e) and (h).

1. $[\![s]\!]^{\mathcal{T}}_{\eta} \in_{\mathcal{T}} [\![t]\!]^{\mathcal{T}}_{\eta} = s_\eta \in_{\mathcal{T}} b_\eta = S(s_\eta \in t_\eta) = [\![s \in t]\!]^{\mathcal{T}}_{\eta}$.

2. Let $s$ be a closed term. Then, $s \in_{\mathcal{T}} [\![\{x : A\}]\!]^{\mathcal{T}}_{\eta} = s \in_{\mathcal{T}} \{x : A\}_\eta = S(s \in \{x : A\}_\eta)$. On the other hand, $[\![A]\!]^{\mathcal{T}}_{\eta[x \mapsto s]} = S(A_{\eta[x \mapsto s]})$. But, $\vdash s \in \{x : A\}_\eta, \Gamma$ is provable if and only if $\vdash A_{\eta[x \mapsto s]}, \Gamma$ is provable. Hence $S(s \in \{x : A\}_\eta) = S(A_{\eta[x \mapsto s]})$.

3. For formulas, we have $[\![A]\!]^{\mathcal{T}}_{\eta[x \mapsto [\![s]\!]^{\mathcal{T}}_{\eta}]} = S(A_{\eta[x \mapsto [\![s]\!]^{\mathcal{T}}_{\eta}]}) = S(A_{\eta[x \mapsto s\,\eta]}) = S(A[s/x]_\eta) = [\![A[s/x]]\!]^{\mathcal{T}}_{\eta}$. For variables, this is immediate by definition. For abstraction terms, we have

$$\begin{aligned} [\![\{y : A\}]\!]^{\mathcal{T}}_{\eta[x \mapsto [\![s]\!]^{\mathcal{T}}_{\eta}]} &= \{y : A\}_{\eta[x \mapsto [\![s]\!]^{\mathcal{T}}_{\eta}]} \\ &= \begin{cases} \{y : A[s/x]\}_\eta = [\![\{y : A\}[s/x]]\!]^{\mathcal{T}}_{\eta} & \text{if } x \neq y \\ \{y : A\}_\eta = [\![\{y : A\}[s/x]]\!]^{\mathcal{T}}_{\eta} & \text{if } x = y \end{cases} \end{aligned}$$

∎

**Theorem 13** *Let $A$ be a closed formula of SIM. If $A$ is valid, then $\vdash A$ is provable in SIM.*

**Proof**
Suppose $A$ is valid. Then, $1_{\mathcal{T}} \in [\![A]\!]^{\mathcal{T}} = S(A)$. Hence $\vdash A$ is provable in SIM. ∎

# 3 Stratified models of linear set theory

We now give another type of models of linear set theory, in which we assign different interpretations to different occurrences of terms and formulas. As a result of this decision, the models become very simple, although we have a certain drawback as well. Consider, for example, the rule of inference for the additive conjunction in linear logic:

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma}$$

In this rule, the occurrences of the formulas $\Gamma$ in the upper sequents are identified in the lower sequent. Since the two occurrences of the same formula may have different interpretations in our model, this identification is not easily justified. Similar identification is made in the comprehension rule. If the term $s$ has more than two occurrences in the formula $A[s/x]$, then we seem to be identifying the ocurrences of $s$ when we obtain $s \in \{x : A\}$ by the unrestricted comprehension. For this reason, the system SLIM of linear set theory which we consider below is formulated in the multiplicative fragment of linear logic and the comprehension rule in SLIM is strict (or linear), *i.e.*, restricted to the cases where $A[s/x]$ contains at most one occurrences of $s$.

**Definition 14** *Let $A$ be a set. The terms and formulas of SLIM(A) are defined by simultaneous induction as follows:*

1. *Variables $x$, $y$, $z$, ... are terms;*
2. *Costants $\underline{a}$, $\underline{b}$, $\underline{c}$ ... for each element of $A$ are terms;*
3. *If $s$ and $t$ are terms, then $s \in t$ and $s \notin t$ are formulas which are atomic;*
4. *If $A$ is a formula with at most one occurrence of the variable $x$ in it, then $\{x : A\}$ is a term;*
5. *If $A$ and $B$ are formulas, then $A \otimes B$ and $A \parr B$ are formulas.*

The duals $A^{\perp}$ are defined in the standard manner. The axioms and the rules of inference of SLIM(A) are given as follows.

**Axioms:**

$$\vdash s \in t, s \notin t$$

**The rules of inference:**

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}$$

$$\frac{\vdash A[s/x], \Gamma}{\vdash s \in \{y : A[y/x]\}, \Gamma} \qquad \frac{\vdash A[s/x], \Gamma}{\vdash s \notin \{y : A[y/x]^{\perp}\}, \Gamma}$$

where the term $s$ occurs at most once in the formula $A[s/x]$, and $y$ is a fresh variable.

$$\frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

**Proposition 15** *SLIM(A) allows cut-elimination.*

**Proof**
By induction on the size of proofs. ∎

We can assign ranks to each occurrences of terms in a given proof as follows. From now on, we write $\boldsymbol{e}$ in the boldface letter for an occurrence of the term or formula $e$.

**Definition 16** *Let $\pi$ be a proof in SLIM(A). We write $Var_{\pi}$ for the set of all the variables within the axioms in $\pi$. Then a function $\eta : Var_{\pi} \to \omega$ is called an initial rank assignment of $\pi$.*

**Definition 17** *Let $\pi$ be a proof in SLIM(A) and $\eta$ be an initial rank assignment of $\pi$. Then, the rank $\rho_\eta(\boldsymbol{s})$ of the occurrences $\boldsymbol{s}$ of terms $s$ in $\pi$ with respect to $\eta$ are inductively defined as follows:*

1. *For the constants $\underline{a}$, we have $\rho_\eta(\boldsymbol{a}) = 0$.*
2. *For the ocuurences of terms within the axioms in $\pi$,*
   (a) $\rho_\eta(\boldsymbol{x}) = \eta(x)$
   (b) $\rho_\eta(\{\boldsymbol{x} : \boldsymbol{A}\}) = \eta(x) + 1$.
3. *For the occurrences $\{\boldsymbol{y} : \boldsymbol{A}[\boldsymbol{y}/\boldsymbol{x}]\}$ or $\{\boldsymbol{y} : \boldsymbol{A}[\boldsymbol{y}/\boldsymbol{x}]^\perp\}$ of the terms created by a rule of inference from the occurrence $\boldsymbol{A}[\boldsymbol{s}/\boldsymbol{x}]$,*
   (a) $\rho_\eta(\boldsymbol{y}) = \rho_\eta(\boldsymbol{s})$
   (b) $\rho_\eta(\{\boldsymbol{y} : \boldsymbol{A}[\boldsymbol{y}/\boldsymbol{x}]\}) = \rho_\eta(\boldsymbol{s}) + 1$,
   (c) $\rho_\eta(\{\boldsymbol{y} : \boldsymbol{A}[\boldsymbol{y}/\boldsymbol{x}]^\perp\}) = \rho_\eta(\boldsymbol{s}) + 1$.

*Note that $\rho_\eta(\boldsymbol{x})$ does not depend on the choice of an occurrence of the variable $x$ in a given proof. We therefore simply write $\rho_\eta(x)$ for $\rho_\eta(\boldsymbol{x})$.*

We now define the universe in which we interpret the ocuurences of terms of SLIM(A).

**Definition 18** *Let $A$ be a set. The hereditarily finite extension of $A$, denoted $V_\omega(A)$, is defined inductively as follows:*

1. $V_0(A) =_{def} A$
2. $V_{n+1}(A) =_{def} \wp V_n(A) \cup V_n(A)$
3. $V_\omega(A) =_{def} \bigcup_{n\in\omega} V_n(A)$

**Definition 19** *A value assignment $\nu$ with respect to the rank assignment $\eta$ is the function $\nu : Var \to V_\omega(A)$ which respects ranks,* i.e., *$\nu(x) \in V_{\rho_\eta(x)}(A)$ for every variable $x$.*

**Definition 20** *A stratified model $\mathcal{M}$ of SLIM(A) is a pair $(V_\omega(A), \{a_i\})$ with $a_i \in A$ for each constant $\underline{a}_i$ of SIM(A).*

From now on, we regard the terms $\{x : A\}$ and $\{y : A\}$ as different terms. By this new convention, every occurrence of a term has the same rank. Therefore, we write $\rho_\eta(s)$ for $\rho_\eta(\boldsymbol{s})$.

**Definition 21** *The valuation $[\![\ ]\!]_\nu^{\mathcal{M}}$ of terms and formulas with respect to $\mathcal{M}$ and $\nu$ is defined inductively as follows:*

1. $[\![x]\!]^{\mathcal{M}}_{\nu} = \nu(x)$
2. $[\![\underline{a}_i]\!]^{\mathcal{M}}_{\nu} = a_i$
3. *Let* $s$ *be* $\{y : A[y/x]\}$. *Then,*

$$[\![\{y : A[y/x]\}]\!]^{\mathcal{M}}_{\nu} = \{u \in V_{\rho_\eta(s)-1}(A) : [\![A[y/x]]\!]^{\mathcal{M}}_{\nu[y\mapsto u]} = t\}$$

4. $[\![s \in t]\!]^{\mathcal{M}}_{\nu} = \begin{cases} t & \text{if } [\![s]\!]^{\mathcal{M}}_{\nu} \in [\![t]\!]^{\mathcal{M}}_{\nu} \\ f & \text{otherwise.} \end{cases}$
5. $[\![s \notin t]\!]^{\mathcal{M}}_{\nu}$, $[\![A \otimes B]\!]^{\mathcal{M}}_{\nu}$ *and* $[\![A \parr B]\!]^{\mathcal{M}}_{\nu}$ *are interpreted as the classical negation, conjunction and disjunction, respectively.*

**Proposition 22** $[\![\ ]\!]^{\mathcal{M}}_{\nu}$ *respects ranks, i.e.,* $[\![s]\!]^{\mathcal{M}}_{\nu} \in V_{\rho_\eta(s)}(A)$.

**Proof**
We only need to check the case where the term $s$ is $\{y : A[y/x]\}$. But, $[\![t]\!]^{\mathcal{M}}_{\nu} \subset V_{\rho_\eta(s)-1}(A)$. Hence, $[\![s]\!]^{\mathcal{M}}_{\nu} \in V_{\rho_\eta(s)}(A)$. ∎

**Lemma 23** *Let* $e$ *be a formula or term,* $s$ *a term and* $x$ *a free variable in* $e$. *Then,* $[\![e[s/x]]\!]^{\mathcal{M}}_{\nu} = [\![e]\!]^{\mathcal{M}}_{\nu[x\mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]}$, *where we assume the renaming of bound variables with the same rank to avoid the variable conflict.*

**Proof**
This is shown by induction on the construction of terms and formulas as ususal. The only interesting case is when $e$ is $\{y : A\}$.

$$\begin{aligned}
[\![\{y : A\}]\!]^{\mathcal{M}}_{\nu[x\mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]} &= \{u \in V_{\rho_\eta(e)-1}(A) : [\![A]\!]^{\mathcal{M}}_{\nu[y\mapsto u, x\mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]} = t\} \\
&= \{u \in V_{\rho_\eta(e)-1}(A) : [\![A]\!]^{\mathcal{M}}_{\nu[y\mapsto u, x\mapsto [\![s]\!]^{\mathcal{M}}_{\nu[y\mapsto u]}]} = t\} \\
&= \{u \in V_{\rho_\eta(e)-1}(A) : [\![A[s/x]]\!]^{\mathcal{M}}_{\nu[y\mapsto u]} = t\} \\
&= [\![\{y : A[s/x]\}]\!]^{\mathcal{M}}_{\nu}
\end{aligned}$$

∎

**Theorem 24** *Suppose*

1. $\pi$ *is a cut-free proof of the sequent* $\vdash A_1, \ldots A_n$,
2. $\eta$ *is an initial rank assignment of* $\pi$,
3. $\nu$ *is a value assignment with respect to* $\eta$,
4. $\mathcal{M}$ *is a stratified model.*

*Then* $[\![A_i]\!]^{\mathcal{M}}_{\nu} = t$ *for at least one of* $A_1, \ldots A_n$.

**Proof**
Since the axioms and the rules of inference for $\otimes$ and $\parr$ are classicaly sound, it suffices to verify the soundness of the two rules of inference for the set abstraction. However,

$$\begin{aligned}
[\![s \in \{y : A[y/x]\}]\!]^{\mathcal{M}}_{\nu} = t \quad &\Leftrightarrow \quad [\![s]\!]^{\mathcal{M}}_{\nu} \in [\![\{y : A[y/x]\}]\!]^{\mathcal{M}}_{\nu} \\
&\Leftrightarrow \quad [\![A[y/x]]\!]^{\mathcal{M}}_{\nu[y \mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]} = t \\
&\Leftrightarrow \quad [\![A[s/x]]\!]^{\mathcal{M}}_{\nu} = t
\end{aligned}$$

$$\begin{aligned}
[\![s \notin \{y : A[y/x]\}]\!]^{\mathcal{M}}_{\nu} = f \quad &\Leftrightarrow \quad [\![s]\!]^{\mathcal{M}}_{\nu} \in [\![\{y : A[y/x]\}]\!]^{\mathcal{M}}_{\nu} \\
&\Leftrightarrow \quad [\![A[y/x]]\!]^{\mathcal{M}}_{\nu[y \mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]} = t \\
&\Leftrightarrow \quad [\![A[y/x]^{\perp}]\!]^{\mathcal{M}}_{\nu[y \mapsto [\![s]\!]^{\mathcal{M}}_{\nu}]} = f \\
&\Leftrightarrow \quad [\![A[s/x]^{\perp}]\!]^{\mathcal{M}}_{\nu} = f
\end{aligned}$$

∎

# 4 The directions of further research

The semantics of set theory based on linear logic is much less developed than its syntax. This paper is intended only as a preparatory work for the more extensive study. Some of the problems are as follows.

1. The Scott style construction of models: Find the model $M$ of linear logic and the partial order on $M$ such that the solution to the equation $U \cong [U \to M]$ gives the model of linear set theory.

2. The completeness of stratified models: Formulate and prove the completeness of SLIM with respect to stratified models.

3. The comparison of SLIM with simple type theory: Establish the relationship between SLIM and simple type theory *via* stratified models.

4. The consistency proof of linear set theory with a weakened extensionality: Construct models which validates the extensionality principle to a certain extent.

The last problem is particularly interesting. It is known that linear set theory together with the standard extensionality principle is inconsistent [4, 9, 6]. Some of the authors, however, made conjectures that certain weakened versions of extensionality are safe [5, 6]. For SLIM, we can indeed show the consistency even with the standard extensionality by the proof-theoretic method, provided that the substitution under equality is also strict (linear). If the comprehension is not strict, however, the proof-theoretic technique does not seem to work, and the semantic method may be required to prove those conjectures.

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