A linear conservative extension of Zermelo-Fraenkel set theory

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Abstract

In this paper, we develop the system LZF of set theory with the unrestricted comprehension in full linear logic and show that LZF is a conservative extension of ZF^- *i.e.*, the Zermelo-Fraenkel set theory without the axiom of regularity. We formulate LZF as a sequent calculus with abstraction terms and prove the partial cut-elimination theorem for it. The cut-elimination result ensures the subterm property for those formulas which contain only terms corresponding to sets in ZF^-. This implies that LZF is a conservative extension of ZF^- and therefore the former is consistent relative to the latter.

1 Introduction

In this paper, we develop the system LZF of set theory with the unrestricted comprehension in full linear logic. The fine distinction provided by linear logic between contractible and non-contractible formulas makes it possible to have more sets than in classical set theory. For the sake of simplicity, LZF is built on top of ZF^- *i.e.*, ZF without the axiom of regularity. It should be noted, however, that we can simply add the axiom of regularity as an axiom of our system without changing much of our argument.

We formulate LZF as one-sided sequent calculus with abstraction terms and prove the partial cut-elimination theorem for it. The cut-elimination result ensures the subterm property for those formulas which contain only terms corresponding to sets in ZF^-. This implies that LZF is a conservative extension of ZF^- and therefore the former is consistent relative to the latter.

The proof of the cut-elimination theorem relies on the fact that the cut-elimination procedure reduces the size of a proof if the contraction rule is not

used in the original proof. This fact was observed by various authors and has been used to prove the cut-elimination theorems for a variety of systems [2, 3, 4, 5, 6, 7, 8, 9].

For the basics of linear logic, we refer to Girard's original paper [2], and Avron's work [1] for more accessible exposition. We use Gentzen–type sequent calculus formulation since the treatment of multiplicative conjunction and exponentials in natural deduction is not so straightforward. Furthermore, the one-sided version of sequent calculus is used since it makes our system much more economical.

One of the intended application areas of our study is the foundation of category theory and universal algebra. For this purpose, however, the further extension of LZF is required.

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2 Preliminary

We review some of the basics about ZF and the translation of classical logic into linear logic.

2.1 Axioms of ZF

The axiomatic system of ZF consists of:

(Ext) $\forall u \forall v[\forall w(w \in u \leftrightarrow w \in v) \rightarrow u = v]$

(Empty) $\exists u \forall v(v \notin u)$

(Pair) $\forall u_1 \forall u_2 \exists v \forall w(w \in v \leftrightarrow w = u_1 \vee w = u_2)$

(Union) $\forall u \exists v \forall w[w \in v \leftrightarrow \exists w_1(w \in w_1 \wedge w_1 \in u)]$

(Power) $\forall u \exists v \forall w[w \in v \leftrightarrow \forall w_1(w_1 \in w \rightarrow w_1 \in u)]$

(Separation) $\forall v \exists w \forall w_1(w_1 \in w \leftrightarrow w_1 \in v \wedge A(w_1))$

(Infinity) $\exists u[\emptyset \in u \wedge \forall v(v \in u \rightarrow v \cup \{v\} \in u)]$

(Regularity) $\forall u[u \neq \emptyset \rightarrow \exists v(v \in u \wedge v \cap u = \emptyset)]$

(Replacement) $\forall u[\forall w(w \in u \rightarrow \forall w_1 \forall w_2(F(w, w_1) \wedge F(w, w_2) \rightarrow w_1 = w_2)) \rightarrow \exists v \forall w(w \in v \leftrightarrow \exists w_1(w_1 \in u \wedge F(w_1, w)))]$

Our first task is to reformulate ZF in the language of linear logic with the abstraction operator. For the sake of simplicity, we will omit the axiom of regularity. Hence our target theory is ZF^-.

In ZF, the use of an abstraction term is acceptable only when we can show the existence and uniqueness of the set denoted by the term. In the cases of (Empty), (Pair), (Union), (Power) and (Separation), we simply introduce the abstraction terms for the sets which are said to exist by the axioms.

It is not immediately clear how we should introduce the terms for the sets given by (Infinity) and (Replacement), since they do not have the form:

$$\exists u \forall w (w \in u \leftrightarrow \varphi(w))$$

However we have the equivalent formulations of those axioms as follows:

(Infinity*) $\exists u \forall w [w \in u \leftrightarrow \forall v (Ind(v) \rightarrow w \in v)]$

(Replacement*) $\forall u \exists v \forall w [w \in v \leftrightarrow \exists w_1 (w_1 \in u \wedge Fun(F, w_1) \wedge F(w_1, w))]$

where

- $Ind(v) \equiv \emptyset \in v \wedge \forall w_1 (w_1 \in v \rightarrow w_1 \cup \{w_1\} \in v)$
- $Fun(F, w_1) \equiv \forall u_1 \forall u_2 (F(w_1, u_1) \wedge F(w_1, u_2) \rightarrow u_1 = u_2)$

The equivalence between (Infinity) and (Infinity*) is proved as follows:

1. (Infinity)$\Rightarrow$(Infinity*)
   This is immediate by the usual definition of ω.

2. (Infinity*)$\Rightarrow$(Infinity)
   Let ω be the witness of (Infinity*).

   - Suppose $Ind(v)$. Then $\emptyset \in v$. Hence, $\forall v (Ind(v) \rightarrow \emptyset \in v)$. Therefore $\emptyset \in \omega$.
   - Suppose $w_1 \in \omega$. Then $\forall v (Ind(v) \rightarrow w_1 \in v)$. In particular, $Ind(v) \rightarrow w_1 \in v$. Assume $Ind(v)$. Then $w_1 \in v$. Furthermore, $w_1 \in v \rightarrow w_1 \cup \{w_1\} \in v$. Hence $w_1 \cup \{w_1\} \in v$. Therefore $\forall v (Ind(v) \rightarrow w_1 \cup \{w_1\} \in v)$. Hence $w_1 \cup \{w_1\} \in \omega$.

The equivalence between (Replacement) and (Replacement*) is shown as follows:

1. (Replacement)$\Rightarrow$(Replacement*)
   Define the subset $u^* = \{w \in u : Fun(F, w)\}$. Then the set obtained by (Replacemnt) from u^* is the witness of (Replacement*).

2. (Replacement*)$\Rightarrow$(Replacement)
   Assume $\forall w (w \in u \rightarrow Fun(F, w))$. Then the set obtained by (Replacement*) from u witnesses (Replacement).

Since the uniqueness is immediate by the extensionality, we are then justified in using the following abstraction terms:

- $\emptyset \equiv \{w : \bot\}$
- $\mathcal{P}(u_1, u_2) \equiv \{w : w = u_1 \vee w = u_2\}$
- $\bigcup(u) \equiv \{w : \exists v(w \in v \wedge v \in u)\}$
- $\wp(u) \equiv \{w : \forall v(v \in w \rightarrow v \in u)\}$
- $\mathcal{S}(u, A) \equiv \{w : w \in u \wedge A(w)\}$
- $\omega \equiv \{w : \forall v(Ind(v) \rightarrow w \in v)\}$
- $\Re(u, F) \equiv \{w : \exists v(v \in u \wedge Fun(F, v) \wedge F(v, w))\}$

2.2 Translation of classical logic into linear logic

Our next aim is to embed the terms and formulas of classical logic into those of linear logic. For this purpose, we use the following translation $(\ )^+$ proposed by Girard [2]:

- $A^+ \equiv\ ?\,!\,A$ where A is atomic
- $(A \vee B)^+ \equiv A^+ \parr B^+$
- $(\neg A)^+ \equiv\ ?\,(A^{+\bot})$
- $(\forall x A)^+ \equiv\ ?\,!\bigwedge x.A^+$

Accordingly, we have:

- $(A \wedge B)^+ \equiv (\neg(\neg A \vee \neg B))^+ \equiv\ ?\,(\,!\,A^+ \otimes\,!\,B^+)$
- $(A \rightarrow B)^+ \equiv (\neg A \vee B)^+ \equiv\ ?\,(A^{+\bot}) \parr B^+ \equiv\,!\,A^+ \multimap B^+$
- $(\exists x A)^+ \equiv (\neg\forall x \neg A)^+ \equiv\ ?\,!\,?\bigvee x\,!\,A^+ \equiv\ ?\bigvee x.\,!\,A^+$

Note that we are using the same propositional letters for classical and linear logics. The specific choice of translation has no particular importance for our purpose and other translations should work as well.

For the translation of schematic expressions, it is convenient to introduce place-holders φ and $\alpha, \ldots$, for a formula and terms, respectively. Variables and place-holders for terms are unchanged under the translation. Then we can define the translation of the constants and abstraction schemas straightforwardly:

- $\emptyset^+ \equiv \{w : \bot\}$
- $\mathcal{P}^+(\alpha_1, \alpha_2) \equiv \{w : (w = \alpha_1 \vee w = \alpha_2)^+\}$

- $\bigcup^+(\alpha) \equiv \{w : [\exists v(w \in v \wedge v \in \alpha)]^+\}$
- $\wp^+(\alpha) \equiv \{w : [\forall v(v \in w \rightarrow v \in \alpha)]^+\}$
- $\mathcal{S}^+(\alpha, \varphi) \equiv \{w : [w \in \alpha \wedge \varphi(w)]^+\}$
- $\omega^+ \equiv \{w : [\forall v(Ind(v) \rightarrow w \in v)]^+\}$
- $\Re^+(\alpha, \varphi) \equiv \{w : [\exists v(v \in \alpha \wedge Fun(\varphi, v) \wedge F(v, w))]^+\}$

3 The linear set theory LZF

3.1 The language of LZF

In the language of LZF , terms are variables and abstraction terms constructed freely from formulas. There are two subclasses of terms: Z-terms and S-terms. Z-terms are terms which correspond to sets in ZF^- and S-terms are terms obtained by abstraction over exponential-free formulas. Roughly speaking, the terms of LZF in general correspond to *classes* in GB. Among them, Z-terms are also *sets*. S-terms comprise all sets and some of the *proper classes* such that the arguments about them should not use the contraction rule.

Definition 1 *We define terms and formulas of LZF inductively as follows:*

- *Z-variables, denoted* u, v, w, $\ldots$, *are terms;*
- *S-variables, denoted* $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\ldots$, *are terms;*
- *If* A *is a formula, then* $\{u : A\}$ *and* $\{\mathbf{x} : A\}$ *are terms;*
- *If* s *and* t *are terms, then* $s \in t$, $s \notin t$, $s = t$ *and* $s \neq t$ *are formulas;*
- $\mathbf{1}$, $\bot$, $\top$ *and* $\mathbf{0}$ *are formulas;*
- *If* A *is a formula, then* $\bigwedge \mathbf{x}.A$, $\bigvee \mathbf{x}.A$, $\bigwedge u.A$, $\bigvee u.A$, $!A$ *and* $?A$ *are formulas;*
- *If* A *and* B *are formulas, then* $A \otimes B$, $A \parr B$, $A \& B$ *and* $A \oplus B$ *are formulas.*

Constants and formulas of the forms $s \in t$, $s \notin t$, $s = t$ *and* $s \neq t$ *are called atomic.*

Definition 2 *The linear negation of formulas are defined in the standard way:*

- $\mathbf{1}^{\perp} \equiv \perp$
- $\top^{\perp} \equiv \mathbf{0}$
- $(s \in t)^{\perp} \equiv s \notin t$
- $(s = t)^{\perp} \equiv s \neq t$
- $(A \otimes B)^{\perp} \equiv A^{\perp} \parr B^{\perp}$
- $(A \& B)^{\perp} \equiv A^{\perp} \oplus B^{\perp}$
- $(\bigwedge \mathbf{x}.A)^{\perp} \equiv \bigvee \mathbf{x}.A^{\perp}$
- $(\bigwedge u.A)^{\perp} \equiv \bigvee u.A^{\perp}$
- $(\,!\,A)^{\perp} \equiv \,?\,A^{\perp}$
- $\perp^{\perp} \equiv \mathbf{1}$
- $\mathbf{0}^{\perp} \equiv \top$
- $(s \notin t)^{\perp} \equiv s \in t$
- $(s \neq t)^{\perp} \equiv s = t$
- $(A \parr B)^{\perp} \equiv A^{\perp} \otimes B^{\perp}$
- $(A \oplus B)^{\perp} \equiv A^{\perp} \& B^{\perp}$
- $(\bigvee \mathbf{x}.A)^{\perp} \equiv \bigwedge \mathbf{x}.A^{\perp}$
- $(\bigvee u.A)^{\perp} \equiv \bigwedge u.A^{\perp}$
- $(\,?\,A)^{\perp} \equiv \,!\,A^{\perp}$

Definition 3 *Z-terms and Z-formulas are defined as follows:*

- *Z-variables are Z-terms;*
- $\emptyset^{+}$ *and* ω^{+} *are Z-terms;*
- *If* s_1 *and* s_2 *are Z-terms, then* $\mathcal{P}^{+}(s_1, s_2)$ *is a Z-term;*
- *If* s *is a Z-term, then* $\bigcup^{+}(s)$ *and* $\wp^{+}(s)$ *are Z-terms;*
- *If* s *is a Z-term and* A *is a Z-formula, then* $\mathcal{S}^{+}(s, A)$ *is a Z-term;*
- *If* s *is a Z-term and* F *is a Z-formula, then* $\Re^{+}(s, F)$ *is a Z-term;*
- *Z-formulas are those formulas in which only Z-terms occur;*

where the abstraction is construed as over Z-variables.

Definition 4 *S-terms and S-formulas are defined as follows:*

- *S-variables are S-terms;*
- *Z-terms are S-terms;*
- *If* A *is an S-formula, then* $\{\mathbf{x} : A\}$ *is an S-term;*
- *If* s *and* t *are S-terms, then* $s \in t$ *and* $s \notin t$ *are S-formula;*
- $\mathbf{1}$, $\perp$, $\top$ *and* $\mathbf{0}$ *are S-formulas;*
- *If* A *is an S-formula, then* $\bigwedge \mathbf{x}.A$ *and* $\bigvee \mathbf{x}.A$ *are S-formulas;*
- *If* A *and* B *are S-formulas, then* $A \otimes B$, $A \parr B$, $A \& B$ *and* $A \oplus B$ *are S-formulas.*

In short, Z-terms are the translations of ZF terms and S-terms are terms which are built on top of Z-terms without using exponentials. As a result, S-terms contain no occurrence of exponentials except within Z-terms. All the Z-terms are S-terms, but not the other way around.

We denote the classes of Z-terms and S-terms by $\mathcal{ZT}$ and $\mathcal{ST}$, respectively. We call an atomic formula admissible if it is an S-formula or it has the form $s = t$ with any terms s and t. In the axiomatic system of LZF , there is only one critical restriction: *The atomic formulas in the axiom*

$$\vdash s \in t, s \notin t$$

contain only S-terms.

3.2 The axioms and rules of inference

We present the axioms and the rules of inference of LZF. In all the axioms except those for the constants, only admissible formulas may appear. All the rules of inferences are standard except that we have two types of quantifiers and abstraction introduction rules according to the two types of variables. Note that the abstraction terms newly introduced by the rules of inference are not necessarily S-terms. Furthermore we do not impose any restriction on terms in the abstraction with S-variable. In fact terms which are not S-terms can be regarded as "syntactic sugar", which are eliminable if necessary.

In the following, the upper Greek letter Γ, Δ, ..., denote multi-sets of formulas. Hence, the exchange rule is built in from the beginning. We also write $FV(\Gamma)$ and $FV(s)$ to denote the set of free variables in Γ and s, respectively.

Axioms:

$$(I) \quad \vdash A, A^{\perp} \quad \textit{where } A \textit{ is admissible.} \qquad (ref) \quad \vdash s = s$$

$$(sub) \quad \vdash s \neq t, A, A[t/s]^{\perp} \quad \textit{where } s,\ t \in \mathcal{ZT} \textit{ and } A \textit{ is an atomic Z-formula.}$$

$$(\mathbf{1}) \quad \vdash \mathbf{1} \qquad\qquad (\top) \quad \vdash \top, \Gamma$$

Rules of inference

$$(\otimes) \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad\qquad (\parr) \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

$$(\&) \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \qquad (\oplus 1) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad (\oplus 2) \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

$$(\bigwedge Z) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge u.A} \quad \textit{with } u \notin FV(\Gamma) \qquad (\bigvee Z) \quad \frac{\vdash \Gamma, A[s/u]}{\vdash \Gamma, \bigvee u.A} \quad \textit{with } s \in \mathcal{ZT}$$

$$(\bigwedge S) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge \mathbf{x}.A} \quad \textit{with } \mathbf{x} \notin FV(\Gamma) \qquad (\bigvee S) \quad \frac{\vdash \Gamma, A[s/\mathbf{x}]}{\vdash \Gamma, \bigvee \mathbf{x}.A} \quad \textit{with } s \in \mathcal{ST}$$

$$(\in Z) \quad \frac{\vdash \Gamma, A[s/u]}{\vdash \Gamma, s \in \{u : A\}} \quad \textit{with } s \in \mathcal{ZT} \qquad (\notin Z) \quad \frac{\vdash \Gamma, A[s/u]^{\perp}}{\vdash \Gamma, s \notin \{u : A\}} \quad \textit{with } s \in \mathcal{ZT}$$

$$(\in S) \quad \frac{\vdash \Gamma, A[s/\mathbf{x}]}{\vdash \Gamma, s \in \{\mathbf{x} : A\}} \qquad (\notin S) \quad \frac{\vdash \Gamma, A[s/\mathbf{x}]^{\perp}}{\vdash \Gamma, s \notin \{\mathbf{x} : A\}}$$

$$(W\,?) \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?\,A} \qquad (D\,?) \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?\,A}$$

$$(C\,?) \quad \frac{\vdash \Gamma, ?\,A, ?\,A}{\vdash \Gamma, ?\,A} \qquad (!) \quad \frac{\vdash ?\,\Gamma, A}{\vdash ?\,\Gamma, !\,A}$$

$$(ext) \quad \frac{\vdash ?\,(u \in s), ?\,(u \notin t), \Gamma \quad \vdash ?\,(v \notin s), ?\,(v \in t), \Delta}{\vdash s = t, \Gamma, \Delta} \quad \begin{array}{l} \textit{with } s, t \in \mathcal{ZT} \textit{ and} \\ u, v \notin FV(\Gamma) \cup FV(s) \cup FV(t) \end{array}$$

$$(\perp) \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \qquad (cut) \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}$$

3.3 The augumented system LZF$^+$

In the proof of the cut-elimination theorem, it is useful to consider the augumented system LZF$^+$ which has the extra rule combining contraction and cut:

$$(?\,cut) \quad \frac{\vdash \Gamma, ?\,A^k \quad \vdash \Delta, !\,A^{\perp}}{\vdash \Gamma, \Delta}$$

where $1 \leq k$ and $?\,A^k$ denotes k occurrences of the formula $?\,A$. When $k = 1$, $(?\,cut)$ can be regarded as the ordinary (cut). Clearly $(?\,cut)$ is a derived rule of LZF so that:

Proposition 5 *If $\vdash \Gamma$ is provable in LZF$^+$, then it is also provable in LZF.*

Proof
Simply replace any use of $(\,?\,cut)$ by

$$\dfrac{\dfrac{\vdash \Gamma,\, ?\, A^k}{\vdash \Gamma,\, ?\, A} \quad \vdash \Delta,\, !\, A^{\perp}}{\vdash \Gamma, \Delta}$$

where we first apply the contraction rules $k-1$ times and then use the ordinary cut. ∎

4 Reduction rules

In our cut-elimination procedure, the cuts are eliminated by consecutive applications of one-step reductions of proofs. Consider the proof:

$$\pi \equiv \dfrac{\begin{array}{cc}\vdots\pi_1 & \vdots\pi_2 \\ \vdash \Gamma, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ (cut)$$

or

$$\pi \equiv \dfrac{\begin{array}{cc}\vdots\pi_1 & \vdots\pi_2 \\ \vdash \Gamma,\, ?\, A^k & \vdash \Delta,\, !\, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ (\,?\,cut)$$

For (cut), we have two kinds of reduction rules:

1. Symmetric reductions: A and $A^{\perp}$ are the main formulas of π_1 and π_2, respectively. This case includes when one of π_i's is an axiom.
2. Permutative reductions: A or $A^{\perp}$ is not the main formula.

For $(\,?\,cut)$, we need to divide permutative reductions into three kinds so that we have:

1. Symmetric reductions: both of $?\,A^k$ and $!\,A^{\perp}$ are the main formulas of π_1 and π_2, including the case where one of π_i's is an axiom.
2. Permutative reductions with !-branch: $!\,A^{\perp}$ is not the main formula of π_2.
3. Permutative reductions with ?-branch: $!\,A^{\perp}$ is the main formula of π_2 and $?\,A$ is not the main formula of π_1.

Except for this complication, all the reduction rules are standard.

4.1 Symmetric reductions for cut

4.1.1 $(\otimes/\parr)$

$$\dfrac{\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \quad \dfrac{\vdash \Theta, A^\perp, B^\perp}{\vdash \Theta, A^\perp \parr B^\perp}}{\vdash \Gamma, \Delta, \Theta} \rightsquigarrow \dfrac{\vdash \Delta, B \quad \dfrac{\vdash \Gamma, A \quad \vdash \Theta, A^\perp, B^\perp}{\vdash \Gamma, \Theta, B^\perp}}{\vdash \Gamma, \Delta, \Theta}$$

4.1.2 $(\&/\oplus 1)$

$$\dfrac{\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \quad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, A^\perp \oplus B^\perp}}{\vdash \Gamma, \Delta} \rightsquigarrow \dfrac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}$$

4.1.3 $(\&/\oplus 2)$

$$\dfrac{\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \quad \dfrac{\vdash \Delta, B^\perp}{\vdash \Delta, A^\perp \oplus B^\perp}}{\vdash \Gamma, \Delta} \rightsquigarrow \dfrac{\vdash \Gamma, B \quad \vdash \Delta, B^\perp}{\vdash \Gamma, \Delta}$$

4.1.4 $(\bigwedge Z/\bigvee Z)$

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge uA} \quad \dfrac{\vdash \Delta, A[s/u]^\perp}{\vdash \Delta, \bigvee uA^\perp}}{\vdash \Gamma, \Delta} \rightsquigarrow \dfrac{\vdash \Gamma, A[s/u] \quad \vdash \Delta, A[s/u]^\perp}{\vdash \Gamma, \Delta}$$

4.1.5 $(\bigwedge S/\bigvee S)$

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge \mathbf{x}A} \quad \dfrac{\vdash \Delta, A[s/\mathbf{x}]^\perp}{\vdash \Delta, \bigvee \mathbf{x}A^\perp}}{\vdash \Gamma, \Delta} \rightsquigarrow \dfrac{\vdash \Gamma, A[s/\mathbf{x}] \quad \vdash \Delta, A[s/\mathbf{x}]^\perp}{\vdash \Gamma, \Delta}$$

4.1.6 $(\in Z/\notin Z)$

$$\dfrac{\dfrac{\vdash \Gamma, A[s/u]}{\vdash \Gamma, s \in \{u : A\}} \quad \dfrac{\vdash \Delta, A[s/u]^\perp}{\vdash \Delta, s \notin \{u : A\}}}{\vdash \Gamma, \Delta} \rightsquigarrow \dfrac{\vdash \Gamma, A[s/u] \quad \vdash \Delta, A[s/u]^\perp}{\vdash \Gamma, \Delta}$$

4.1.7 $(\in S/\notin S)$

$$\frac{\dfrac{\vdash \Gamma, A[s/\mathbf{x}]}{\vdash \Gamma, s \in \{\mathbf{x} : A\}} \quad \dfrac{\vdash \Delta, A[s/\mathbf{x}]^{\perp}}{\vdash \Delta, s \notin \{\mathbf{x} : A\}}}{\vdash \Gamma, \Delta} \rightsquigarrow \frac{\vdash \Gamma, A[s/\mathbf{x}] \ \vdash \Delta, A[s/\mathbf{x}]^{\perp}}{\vdash \Gamma, \Delta}$$

4.1.8 $(W\ ?\,/\,!)$

$$\frac{\dfrac{\vdash \Gamma}{\vdash \Gamma, ?\,A} \quad \dfrac{\vdash ?\,\Delta, A^{\perp}}{\vdash ?\,\Delta, !\,A^{\perp}}}{\vdash \Gamma, ?\,\Delta} \rightsquigarrow \frac{\vdash \Gamma}{\vdash \Gamma, ?\,\Delta}$$

4.1.9 $(D\ ?\,/\,!)$

$$\frac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?\,A} \quad \dfrac{\vdash ?\,\Delta, A^{\perp}}{\vdash ?\,\Delta, !\,A^{\perp}}}{\vdash \Gamma, ?\,\Delta} \rightsquigarrow \frac{\vdash \Gamma, A \ \vdash ?\,\Delta, A^{\perp}}{\vdash \Gamma, ?\,\Delta}$$

4.1.10 $(\perp/\mathbf{1})$

$$\frac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \perp} \quad \vdash \mathbf{1}}{\vdash \Gamma} \rightsquigarrow \vdash \Gamma$$

4.1.11 $(*/I)$

$$\frac{\vdash \Gamma, A \ \vdash A, A^{\perp}}{\vdash \Gamma, A} \rightsquigarrow \vdash \Gamma, A$$

4.1.12 $(*/\top)$

$$\frac{\vdash \Gamma, A \ \vdash \top, \Delta, A^{\perp}}{\vdash \top, \Gamma, \Delta} \rightsquigarrow \vdash \top, \Gamma, \Delta$$

4.2 Symmetric reductions for ?cut

4.2.1 $(W\ ?\,/\,!)$–?

$$\frac{\dfrac{\vdash \Gamma, ?\,A^{k}}{\vdash \Gamma, ?\,A^{k+1}} \ \vdash ?\,\Delta, !\,A^{\perp}}{\vdash \Gamma, ?\,\Delta} \rightsquigarrow \frac{\vdash \Gamma, ?\,A^{k} \ \vdash ?\,\Delta, !\,A^{\perp}}{\vdash \Gamma, ?\,\Delta}$$

4.2.2 $(D\ ?\,/\,!)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?A^k, A}{\vdash \Gamma, ?A^{k+1}} \quad \dfrac{\vdash ?\Delta, A^\perp}{\vdash ?\Delta, !A^\perp}}{\vdash \Gamma, ?\Delta} \rightsquigarrow \dfrac{\dfrac{\dfrac{\vdash \Gamma, ?A^k, A \vdash ?\Delta, !A^\perp}{\vdash \Gamma, ?\Delta, A} \quad \vdash ?\Delta, A^\perp}{\vdash \Gamma, ?\Delta, ?\Delta}}{\vdash \Gamma, ?\Delta}$$

4.2.3 $(C\ ?\,/\,!)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?A^k, ?A}{\vdash \Gamma, ?A^k} \quad \vdash ?\Delta, !A^\perp}{\vdash \Gamma, ?\Delta} \rightsquigarrow \dfrac{\vdash \Gamma, ?A^k, ?A \vdash ?\Delta, !A^\perp}{\vdash \Gamma, ?\Delta}$$

4.3 Permutative reductions for cut

4.3.1 Permutative reductions with two-premiss rules

4.3.1.1 $(cut/\otimes)$

$$\dfrac{\vdash \Gamma, C \quad \dfrac{\vdash \Delta, C^\perp, A \vdash \Theta, B}{\vdash \Delta, \Theta, C^\perp, A \otimes B}}{\vdash \Gamma, \Delta, \Theta, A \otimes B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, C \vdash \Delta, C^\perp, A}{\vdash \Gamma, \Delta, A} \quad \vdash \Theta, B}{\vdash \Gamma, \Delta, \Theta, A \otimes B}$$

4.3.1.2 $(cut/\&)$

$$\dfrac{\vdash \Gamma, C \quad \dfrac{\vdash \Delta, C^\perp, A \vdash \Delta, C^\perp, B}{\vdash \Delta, C^\perp, A\&B}}{\vdash \Gamma, \Delta, A\&B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, C \vdash \Delta, C^\perp, A}{\vdash \Gamma, \Delta, A} \quad \dfrac{\vdash \Gamma, C \vdash \Delta, C^\perp, B}{\vdash \Gamma, \Delta, B}}{\vdash \Gamma, \Delta, A\&B}$$

4.3.1.3 (cut/ext)

$$\dfrac{\vdash C, \Gamma \quad \dfrac{\vdash \Sigma_1, C^\perp, \Delta \quad \vdash \Sigma_2, \Theta}{\vdash s = t, C^\perp, \Delta, \Theta}}{\vdash s = t, \Gamma, \Delta, \Theta} \rightsquigarrow \dfrac{\dfrac{\vdash C, \Gamma \quad \vdash \Sigma_1, C^\perp, \Delta}{\vdash \Sigma_1, \Gamma, \Delta} \quad \vdash \Sigma_2, \Theta}{\vdash s = t, \Gamma, \Delta, \Theta}$$

where Σ_1 and Σ_2 denote $?(u \in s)$, $?(u \notin t)$ and $?(v \notin s)$, $?(v \in t)$, respectively.

4.3.1.4 (cut/cut)

$$\dfrac{\vdash \Gamma, C \quad \dfrac{\vdash \Delta, C^\perp, A \vdash \Theta, A^\perp}{\vdash \Delta, \Theta, C^\perp}}{\vdash \Gamma, \Delta, \Theta} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, C \vdash \Delta, C^\perp, A}{\vdash \Gamma, \Delta, A} \quad \vdash \Theta, A^\perp}{\vdash \Gamma, \Delta, \Theta}$$

4.3.1.5 $(cut/\ ?cut)$

$$\dfrac{\vdash \Gamma, C \quad \dfrac{\vdash \Delta, C^{\perp}, ?A^{k} \ \vdash \Theta, !A^{\perp}}{\vdash \Delta, \Theta, C^{\perp}}}{\vdash \Gamma, \Delta, \Theta} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, C \ \vdash \Delta, C^{\perp}, ?A^{k}}{\vdash \Gamma, \Delta, ?A^{k}} \quad \vdash \Theta, !A^{\perp}}{\vdash \Gamma, \Delta, \Theta}$$

4.3.2 Permutative reductions with one-premiss rules

(cut) is freely permutable with all the one-premiss rules except $(!)$. For the permutation with $(!)$, we need to restrict the form of sequents. In fact, this case is subsumed under the permutation rules for $(?\,cut)$ and the rule is given there. Since all the other permutation rules are entirely similar, we only give the rule for $(\parr)$ as example.

4.3.2.1 $(cut/\parr)$

$$\dfrac{\vdash \Gamma, C \quad \dfrac{\vdash \Delta, C^{\perp}, A, B}{\vdash \Delta, C^{\perp}, A \parr B}}{\vdash \Gamma, \Delta, A \parr B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, C \ \vdash \Delta, C^{\perp}, A, B}{\vdash \Gamma, \Delta, A, B}}{\vdash \Gamma, \Delta, A \parr B}$$

4.4 Permutative reductions for ?cut with !-branch

They are entirely similar to the permutation rules for (cut). We distinguish this type of permutation from the others by the suffix !. We only give the permutation rule $(?\,cut/\otimes)$–! as example.

4.4.1 $(?\,cut/\otimes)$–!

$$\dfrac{\vdash \Gamma, ?C^{k} \quad \dfrac{\vdash \Delta, !C^{\perp}, A \vdash \Theta, B}{\vdash \Delta, \Theta, !C^{\perp}, A \otimes B}}{\vdash \Gamma, \Delta, \Theta, A \otimes B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^{k} \ \vdash \Delta, !C^{\perp}, A}{\vdash \Gamma, \Delta, A} \quad \vdash \Theta, B}{\vdash \Gamma, \Delta, \Theta, A \otimes B}$$

4.5 Permutative reductions for ?cut with ?-branch

We indicate this type of permutations with the suffix ?. Note that we are assuming that the cut formula $!C^{\perp}$ is the main formula of the !-branch.

4.5.1 Permutative reductions with two-premiss rules

4.5.1.1 $(\otimes/\ ?\,cut1)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash \Delta, B}{\vdash \Gamma, \Delta, ?C^k, A \otimes B}\ \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, ?\Theta, A}\ \vdash \Delta, B}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B}$$

4.5.1.2 $(\otimes/\ ?\,cut2)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash \Delta, ?C^l, B}{\vdash \Gamma, \Delta, ?C^{k+l}, A \otimes B}\ \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, ?\Theta, A}\ \dfrac{\vdash \Delta, ?C^l, B \vdash ?\Theta, !C^{\perp}}{\vdash \Delta, ?\Theta, B}}{\dfrac{\vdash \Gamma, \Delta, ?\Theta\ ?\Theta, A \otimes B}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B}}$$

The permutative reductions for (ext) is entirely similar to those for $(\otimes)$.

4.5.1.3 $(\&/\ ?\,cut)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash \Gamma, ?C^k, B}{\vdash \Gamma, ?C^k, A\&B}\ \vdash ?\Delta, !C^{\perp}}{\vdash \Gamma, ?\Delta, A\&B} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash ?\Delta, !C^{\perp}}{\vdash \Gamma, ?\Delta, A}\ \dfrac{\vdash \Gamma, ?C^k, B \vdash ?\Delta, !C^{\perp}}{\vdash \Gamma, ?\Delta, B}}{\vdash \Gamma, ?\Delta, A\&B}$$

4.5.1.4 $(cut/\ ?\,cut1)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, ?C^k}\ \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, \Delta, ?\Theta} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, ?\Theta, A}\ \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta, ?\Theta}$$

4.5.1.5 $(cut/\ ?\,cut2)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash \Delta, ?C^l, A^{\perp}}{\vdash \Gamma, \Delta, ?C^{k+l}}\ \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, \Delta, ?\Theta} \rightsquigarrow \dfrac{\dfrac{\vdash \Gamma, ?C^k, A \vdash ?\Theta, !C^{\perp}}{\vdash \Gamma, ?\Theta, A}\ \dfrac{\vdash \Delta, ?C^l, A^{\perp} \vdash ?\Theta, !C^{\perp}}{\vdash \Delta, ?\Theta, A^{\perp}}}{\dfrac{\vdash \Gamma, \Delta, ?\Theta, ?\Theta}{\vdash \Gamma, \Delta, ?\Theta, A \otimes B}}$$

The permutative reductions for $(?\,cut)$ is entirely similar to those for (cut).

4.5.2 Permutative reductions with one-premiss rules

They are entirely similar to the permutations for (cut). Hence we give only the permutation rule with $\parr$ in addition to the permutation with $(!)$ which we promised before.

4.5.2.1 $(\parr / ?cut)$–?

$$\dfrac{\dfrac{\vdash \Gamma, ?C^k, A, B}{\vdash \Gamma, ?C^k, A \parr B} \quad \vdash ?\Delta, !C^{\perp}}{\vdash \Gamma, ?\Delta, A \parr B} \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma, ?C^k, A, B \quad \vdash ?\Delta, !C^{\perp}}{\vdash \Gamma, ?\Delta, A, B} \over \vdash \Gamma, ?\Delta, A \parr B$$

4.5.2.2 $(!/cut)$–?

$$\dfrac{\dfrac{\vdash ?\Gamma, ?C^k, A}{\vdash ?\Gamma, ?C^k, !A} \quad \vdash ?\Delta, !C^{\perp}}{\vdash ?\Gamma, ?\Delta, !A} \quad \rightsquigarrow \quad \dfrac{\vdash ?\Gamma, ?C^k, A \quad \vdash ?\Delta, !C^{\perp}}{\vdash ?\Gamma, ?\Delta, A} \over \vdash ?\Gamma, ?\Delta, !A$$

This completes our reduction rules. The reduction relation between proofs are defined as the transitive and compatible closure of one-step reduction rules. We write $\pi_1 \rightsquigarrow \pi_2$ when π_1 reduces to π_2.

To prove the cut-elimination theorem, we need to introduce three measures: *the logical complexity* $\partial(A)$ of a formula A, *the size* $\sigma(\pi)$ of a proof π and *the abstraction depth* $\delta(A)$ of an occurrence of a formula A.

Definition 6 *The logical complexity $\partial(A)$ of a formula A is defined inductively as follows:*

- *If A is atomic, then $\partial(A) = 1$*
- $\partial(A \otimes B) = \partial(A \parr B) = \partial(A \& B) = \partial(A \oplus B) = max(\partial(A), \partial(B)) + 1$
- $\partial(!A) = \partial(?A) = \partial(A) + 1$

In addition, we define $\partial(\pi)$ to be the supremum of the logical complexity of the cut formulas in a proof π. If there is no cut in π, then $\partial(\pi) = 0$.

Definition 7 *The size $\sigma(\pi)$ of a proof π is defined inductively as follows:*

1. *If π is an axiom, then $\sigma(\pi) = 1$;*
2. *If π is obtained from π_1 by a one-premiss rule, then $\sigma(\pi) = \sigma(\pi_1) + 1$;*
3. *If π is obtained from π_1 and π_2 by $(\otimes)$, (ext), (cut) or $(?cut)$, then $\sigma(\pi) = \sigma(\pi_1) + \sigma(\pi_2)$;*
4. *If π is obtained from π_1 and π_2 by $(\&)$, then $\sigma(\pi) = max(\sigma(\pi_1), \sigma(\pi_2))$.*

The definition of abstraction depth will be given in Section 6. Then, our proof of the cut-elimination theorem has three stages:

1. Eliminate all the cuts except the cuts with atomic formulas. The logical complexity of the cut formulas decreases in this process while the size of a proof may increase. The abstraction depth of the atomic cut formulas *does not increase*.

2. Eliminate the cuts with atomic formulas when they do not invlove axioms. In the process, the abstraction depth of the atomic cut formulas decreases while the logical complexity and the size may increase.

3. Eliminate all the remaining cuts except those with Z-formulas. In the process, the size of the proof decreases while the logical complexity and the abstraction depth of the cut formulas may increase.

In the first and second stages, we are working with LZF^{+} rather than LZF itself.

5 The elimination of non-atomic cuts

Proposition 8 *Consider the proof:*

$$\pi \equiv \dfrac{\begin{array}{cc}\vdots\pi_1 & \vdots\pi_2 \\ \vdash \Gamma, ?A^k & \vdash \Delta, !A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ (?cut)$$

and suppose that the logical complexity $\partial(\pi_1)$ and $\partial(\pi_2)$ of the proofs π_1 and π_2 is strictly less than the logical complexity $\partial(?A)$ of the formula $?A$. Then there exists a proof τ such that $\pi \rightsquigarrow \tau$ and $\partial(\tau) < \partial(?A)$.

Proof
The proof is by induction on $\sigma(\pi_1) + \sigma(\pi_2)$.

Case 1 The last inference of π_2 is not $(!)$.

- **1.1** π_2 is an axiom $(\top)$. Then π reduces to the same type of axiom.
- **1.2** Otherwise, we apply one of the permutative reductions and use the inductive hypothesis.

Case 2 The last inference of π_2 is $(!)$.

- **2.1** π_1 is an axiom $(\top)$. Then π reduces to the same type of axiom.
- **2.2** The main formula of the last inference of π_1 is not one of $?A^k$. Then we apply one of the permutative reductions and use the inductive hypothesis.
- **2.3** The main formula of the last inference of π_1 is one of $?A^k$ and $2 \leq k$. Then we apply one of the symmetric reductions and use the inductive hypothesis.

2.4 The main formula of the last inference is one of $?A^k$ and $k = 1$. Then we apply one of the symmetric reductions.

∎

Proposition 9 *Consider the proof:*

$$\pi \equiv \dfrac{\begin{array}{cc}\vdots\pi_1 & \vdots\pi_2 \\ \vdash \Gamma, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}$$

where the formula A is not of the form $?C$ nor $!C$. Suppose that the logical complexity $\partial(\pi_1)$ and $\partial(\pi_2)$ is strictly less than the logical complexity $\partial(A)$ of the formula A and $2 \leq \partial(A)$. Then there exists a proof τ such that $\pi \rightsquigarrow \tau$ and $\partial(\pi) < \partial(A)$.

Proof
The proof is by induction on $\sigma(\pi_1) + \sigma(\pi_2)$

Case 1 One of π_1 and π_2 is $\top$-axiom. Then π reduces to the same kind of axiom.

Case 2 One of A and $A^{\perp}$ is not the main formula. Then we apply one of the permutative reduction and use the inductive hypothesis.

Case 3 A and $A^{\perp}$ are the main formulas. We then apply the symmetric reduction. Note that we do not use $(\in Z/ \notin Z)$ nor $(\in S/ \notin S)$ since $2 \leq \partial(A)$. Hence the logical complexity decreases.

∎

We can now prove:

Lemma 10 *For any proof π with $2 \leq \partial(\pi)$, there exists a proof τ such that $\pi \rightsquigarrow \tau$ and $\partial(\tau) < \partial(\pi)$.*

Proof
The proof is by induction on $\sigma(\pi)$. Consider the lower-most cut of the maximum logical complexity in π. Apply the inductive hypothesis to both of the branches. Then eliminate the remaining cut by Proposition 8 and 9. ∎

Theorem 11 *For any proof π with $2 \leq \partial(\pi)$, there exists a proof τ of the same sequent such that $\pi \rightsquigarrow \tau$ and $\partial(\tau) \leq 1$.*

Proof
This is immediate from the Lemma 10 by induction on $\partial(\pi)$. ∎

6 The elimination of atomic cuts involving no axioms

So far we have eliminated all the cuts except those with atomic formulas. The next stage of our cut-elimination theorem is to eliminate all the cuts with atomic formulas which do not originate in axioms. For this, we need to introduce the *abstraction depth* of occurrences of formulas. Accordingly, we give the *abstraction index* (i, j) to the cuts with the occurrences of the formulas of abstraction depth i and j.

In contrast to the logical complexity, the abstraction depth of an occurrence of a formula has to do with the place and role of the formula occurrence in a proof. Hence we give the definition of abstraction depth in accordance with the axioms and rules of inferences. In the following, we indicate the assignment of abstraction depth to the occurrences of the formulas in sequents to the right of the axioms and the rules of inference. The abstraction depth increases when one of the abstraction rules is used. When two formula occurrences are combined by one of the inference rules, we take the maximum of the abstraction depths of the two formula occurrences.

First we introduce several useful notations. Let p's and q's be natural numbers.

- $\rho(p_1, \ldots, p_k)$ *denotes* $max\{p_1, \ldots, p_k\}$;
- $\vec{p}$ *denotes the list of* $p_1, \ldots, p_k$ *for some* k;
- $\vec{\rho}(\vec{p}, \vec{q})$ *denotes the list of* $\rho(p_1, q_1), \ldots, \rho(p_k, q_k)$ *where* $\vec{p}$ *and* $\vec{q}$ *have the same length* k.

Axioms:

$$(I)\ \vdash A, A^{\perp}\quad \vdash 0, 0 \qquad\qquad (ref)\ \vdash s = s\quad \vdash 0$$

$$(sub)\ \vdash s \neq t, A, A[t/s]^{\perp}\quad \vdash 0, 0, 0$$

$$(\mathbf{1})\ \vdash \mathbf{1}\quad \vdash 0 \qquad\qquad (\top)\ \vdash \top, A_1, \ldots, A_k\quad \vdash 0, 0, \ldots, 0$$

Rules of inference:

$$(\otimes)\ \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \frac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \rho(i, j)} \qquad (\parr)\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \frac{\vdash \vec{p}, i, j}{\vdash \vec{p}, \rho(i, j)}$$

$$(\&)\ \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \frac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \rho(i, j)} \qquad (\oplus i)\ \frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i}$$

$$(\bigwedge Z)\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge uA}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i} \qquad (\bigvee Z)\ \frac{\vdash \Gamma, A[s/u]}{\vdash \Gamma, \bigvee uA}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i}$$

$$(\bigwedge S)\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \bigwedge \mathbf{x}A}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i} \qquad (\bigvee S)\ \frac{\vdash \Gamma, A[s/\mathbf{x}]}{\vdash \Gamma, \bigvee \mathbf{x}A}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i}$$

$$(\in Z)\ \frac{\vdash \Gamma, A[s/u]}{\vdash \Gamma, s \in \{u : A\}}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i+1} \qquad (\notin Z)\ \frac{\vdash \Gamma, A[s/u]^{\perp}}{\vdash \Gamma, s \notin \{u : A\}}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i+1}$$

$$(\in S)\ \frac{\vdash \Gamma, A[s/\mathbf{x}]}{\vdash \Gamma, s \in \{\mathbf{x} : A\}}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i+1} \qquad (\notin S)\ \frac{\vdash \Gamma, A[s/\mathbf{x}]^{\perp}}{\vdash \Gamma, s \notin \{\mathbf{x} : A\}}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i+1}$$

$$(W\ ?)\ \frac{\vdash \Gamma}{\vdash \Gamma, ?\,A}\ \frac{\vdash \vec{p}}{\vdash \vec{p}, 0} \qquad (D\ ?)\ \frac{\vdash \Gamma, A}{\vdash \Gamma, ?\,A}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i}$$

$$(C\ ?)\ \frac{\vdash \Gamma, ?\,A, ?\,A}{\vdash \Gamma, ?\,A}\ \frac{\vdash \vec{p}, i, j}{\vdash \vec{p}, \rho(i, j)} \qquad (!)\ \frac{\vdash ?\,\Gamma, A}{\vdash ?\,\Gamma, !\,A}\ \frac{\vdash \vec{p}, i}{\vdash \vec{p}, i}$$

$$(ext)\ \frac{\vdash ?\,(v \in s), ?\,(v \notin t), \Gamma \quad \vdash ?\,(v \notin s), ?\,(v \in t), \Delta}{\vdash s = t, \Gamma, \Delta}\ \frac{\vdash i_1, i_2, \vec{p} \quad \vdash j_1, j_2, \vec{q}}{\vdash \rho(i_1, i_2, j_1, j_2), \vec{p}, \vec{q}}$$

$$(\perp)\ \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\quad \frac{\vdash \vec{p}}{\vdash \vec{p}, 0} \qquad (cut)\ \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\quad \frac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (i, j)$$

$$(?\,cut)\ \frac{\vdash \Gamma, ?\,A^{k} \quad \vdash \Delta, !\,A^{\perp}}{\vdash \Gamma, \Delta}\quad \frac{\vdash \vec{p}, \vec{i} \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{i}), j)$$

We denote the abstraction depth of the formula occurrence A by $\delta(A)$. The cut with the index (i, j) is called an (i, j)-cut. Then we define the abstraction depth of a proof π to be $\delta(\pi) = sup\{i \times j : (i, j)\text{-}cut\ in\ \pi\}$. If there is no cut in π, then $\delta(\pi) = 0$.

Consider the occurrence of a formula A in a proof π. By tracing back how the occurrence of A is generated in π, we can talk about its *ancestral formula occurrences*. It should be clear that the abstraction depths of the ancestral formula occurrences are equal or smaller than that of the occurrence A. In the cut-elimination process except that for axioms and weakening, we replace a cut with the occurrence A by other cuts with itself or its ancestral formula occurrences. Therefore $\delta(\pi)$ remains the same or decreases in the cut-elimination process up to the axioms. In this way we can eliminate the cuts with the maximal abstraction depth one by one.

Proposition 12 *None of the reductions except the symmetric reduction $(*/I)$ increase the abstraction depth of a formula occurrence in the conclusion and the abstraction depth of a proof. Furthermore, the symmetric reductions $(\in S/\notin S)$ and $(\in Z/\notin Z)$ strictly decrease the abstraction depth of a proof.*

Proof
The proof is by inspection.

1 Symmetric reductions for cut

1.1 $(\otimes/\parr)$

$$\dfrac{\dfrac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \rho(i,j)} \quad \dfrac{\vdash \vec{r}, k, l}{\vdash \vec{r}, \rho(k,l)}}{\vdash \vec{p}, \vec{q}, \vec{r}}\ (\rho(i,j), \rho(k,l)) \quad \rightsquigarrow \quad \dfrac{\vdash \vec{q}, j \quad \dfrac{\vdash \vec{p}, i \quad \vdash \vec{r}, k, l}{\vdash \vec{p}, \vec{r}, l}\ (i,k)}{\vdash \vec{p}, \vec{q}, \vec{r}}\ (j,l)$$

1.2 $(\&/\oplus)$

$$\dfrac{\dfrac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \rho(i,j)} \quad \dfrac{\vdash \vec{r}, k}{\vdash \vec{r}, k}}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \vec{r}}\ (\rho(i,j), k) \quad \rightsquigarrow \quad \dfrac{\vdash \vec{p}, i \quad \vdash \vec{r}, k}{\vdash \vec{p}, \vec{r}}\ (i,k)$$

1.3 $(\bigwedge Z/\bigvee Z)$, $(\bigwedge S/\bigvee S)$ and $(\,!/D\,?\,)$

$$\dfrac{\dfrac{\vdash \vec{p}, i}{\vdash \vec{p}, i} \quad \dfrac{\vdash \vec{q}, j}{\vdash \vec{q}, j}}{\vdash \vec{p}, \vec{q}}\ (i,j) \quad \rightsquigarrow \quad \dfrac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (i,j)$$

1.4 $(\in Z/\notin Z)$ and $(\in S/\notin S)$

$$\dfrac{\dfrac{\vdash \vec{p}, i}{\vdash \vec{p}, i+1} \quad \dfrac{\vdash \vec{q}, j}{\vdash \vec{q}, j+1}}{\vdash \vec{p}, \vec{q}}\ (i+1, j+1) \quad \rightsquigarrow \quad \dfrac{\vdash \vec{p}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (i,j)$$

1.5 $(\,!/W\,?\,)$

$$\dfrac{\dfrac{\vdash \vec{p}, i}{\vdash \vec{p}, i} \quad \dfrac{\vdash \vec{q}}{\vdash \vec{q}, 0}}{\vdash \vec{p}, \vec{q}}\ (i,0) \quad \rightsquigarrow \quad \dfrac{\vdash \vec{q}}{\vdash 0, \ldots 0, \vec{q}}$$

1.6 $(\bot/\mathbf{1})$

$$\dfrac{\dfrac{\vdash \vec{p}}{\vdash \vec{p}, 0} \quad \vdash 0}{\vdash \vec{p}}\ (0,0) \ \rightsquigarrow\ \vdash \vec{p}$$

1.7 $(*/\top)$

$$\dfrac{\vdash \vec{p}, i \quad \vdash 0, 0, \ldots, 0, 0}{\vdash 0, \vec{p}, 0, \ldots 0}\ (i,0) \ \rightsquigarrow\ \vdash 0, 0, \ldots, 0, 0, \ldots, 0$$

2 Symmetric reductions for ?cut

2.1 $(W\ ?\,/\,!)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}}{\vdash \vec{p}, \vec{m}, 0} \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{m}), j) \ \rightsquigarrow\ \dfrac{\vdash \vec{p}, \vec{m} \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{m}), j)$$

2.2 $(D\ ?\,/\,!)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i}{\vdash \vec{p}, \vec{m}, i} \quad \dfrac{\vdash \vec{q}, j}{\vdash \vec{q}, j}}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{m}), i), j) \ \rightsquigarrow\ \dfrac{\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, i}\ (\rho(\vec{m}), j) \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \vec{q}}}{\vdash \vec{p}, \vec{q}}\ (i,j)$$

2.3 $(C\ ?\,/\,!)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, m_1, \ldots, m_n, \ldots, m_k, i}{\vdash \vec{p}, m_1, \ldots, \rho(m_n, i), \ldots, m_k} \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{m}, i), j) \ \rightsquigarrow\ \dfrac{\vdash \vec{p}, \vec{m}, i \quad \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}}\ (\rho(\vec{m}, i), j)$$

3 Permutative reductions for cut

3.1 $(cut/\otimes)$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l, i \quad \vdash \vec{r}, j}{\vdash \vec{q}, \vec{r}, l, \rho(i,j)}}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}\ (k,l) \ \rightsquigarrow\ \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l, i}{\vdash \vec{p}, \vec{q}, i}\ (k,l) \quad \vdash \vec{r}, j}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}$$

3.2 $(cut/\&)$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l_1, i \quad \vdash \vec{r}, l_2, j}{\vdash \vec{\rho}(\vec{q}, \vec{r}), \rho(l_1, l_2), \rho(i, j)}}{\vdash \vec{p}, \vec{\rho}(\vec{q}, \vec{r}), \rho(i, j)}\ (k, \rho(l_1, l_2)) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l_1, i}{\vdash \vec{p}, \vec{q}, i}\ (k, l_1) \quad \dfrac{\vdash \vec{p}, k \quad \vdash \vec{r}, l_2, j}{\vdash \vec{r}, \vec{p}, j}\ (k, l_2)}{\vdash \vec{p}, \vec{\rho}(\vec{q}, \vec{r}), \rho(i, j)}$$

3.3 (cut/ext)

$$\dfrac{\vdash k, \vec{p} \quad \dfrac{\vdash i_1, i_2, l, \vec{q} \quad \vdash j_1, j_2, \vec{r}}{\vdash \rho(i_1, i_2, j_1, j_2), l, \vec{q}, \vec{r}}}{\vdash \rho(i_1, i_2, j_1, j_2), \vec{p}, \vec{q}, \vec{r}}\ (k, l) \rightsquigarrow \quad \dfrac{\dfrac{\vdash k, \vec{p} \quad \vdash i_1, i_2, l, \vec{q}}{\vdash i_1, i_2, \vec{p}, \vec{q}}\ (k, l) \qquad \vdash j_1, j_2, \vec{r}}{\vdash \rho(i_1, i_2, j_1, j_2), \vec{p}, \vec{q}, \vec{r}}$$

3.4 (cut/cut)

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l, i \quad \vdash \vec{r}, j}{\vdash \vec{q}, \vec{r}, l}\ (i, j)}{\vdash \vec{p}, \vec{q}, \vec{r}}\ (k, l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l, i}{\vdash \vec{p}, \vec{q}, i}\ (k, l) \quad \vdash \vec{r}, j}{\vdash \vec{p}, \vec{q}, \vec{r}}\ (i, j)$$

3.5 $(cut/\parr)$ and $(cut/C\ ?\ )$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l, i, j}{\vdash \vec{r}, l, \rho(i, j)}}{\vdash \vec{p}, \vec{q}, \rho(i, j)}\ (k, l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l, i, j}{\vdash \vec{p}, \vec{q}, i, j}\ (k, l)}{\vdash \vec{p}, \vec{q}, \rho(i, j)}$$

3.6 $(cut/\oplus)$, $(cut/\bigwedge Z)$, $(cut/\bigvee Z)$, $(cut/\bigwedge S)$, $(cut/\bigvee S)$ and $(cut/D\ ?\ )$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l, i}{\vdash \vec{q}, l, i}}{\vdash \vec{p}, \vec{q}, i}\ (k, l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l, i}{\vdash \vec{p}, \vec{q}, i}\ (k, l)}{\vdash \vec{p}, \vec{q}, i}$$

3.7 $(cut/\in Z)$, $(cut/\notin Z)$, $(cut/\in S)$ and $(cut/\notin S)$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l, i}{\vdash \vec{r}, l, i+1}}{\vdash \vec{p}, \vec{q}, i+1}\ (k, l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l, i}{\vdash \vec{p}, \vec{q}, i}\ (k, l)}{\vdash \vec{p}, \vec{q}, i+1}$$

3.8 $(cut/W\ ?\ )$ and $(cut/\bot)$

$$\dfrac{\vdash \vec{p}, k \quad \dfrac{\vdash \vec{q}, l}{\vdash \vec{r}, l, 1}}{\vdash \vec{p}, \vec{q}, 1}\ (k, l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, k \quad \vdash \vec{q}, l}{\vdash \vec{p}, \vec{q}}\ (k, l)}{\vdash \vec{p}, \vec{q}, 1}$$

4 Permutative reductions with !-branch

They are entirely similar to the permutations for (cut) except that we have to replace $\vec{k}$ and $(\rho(\vec{k}), l)$ for k and (k, l), respectively.

5 Permutative reductions with ?-branch

5.1 Permutations with two-premiss rules

5.1.1 $(\otimes/\ ?\,cut1)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \vec{m}, \rho(i,j)}\ \vdash \vec{r}, l}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}\,(\rho(\vec{m}), l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{r}, l}{\vdash \vec{p}, \vec{r}, i}\,(\rho(\vec{m}), l)\ \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}$$

5.1.2 $(\otimes/\ ?\,cut2)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{q}, \vec{n}, j}{\vdash \vec{p}, \vec{q}, \vec{m}, \vec{n}, \rho(i,j)}\ \vdash \vec{r}, l}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}\,(\rho(\vec{m}, \vec{n}), l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{r}, l}{\vdash \vec{p}, \vec{r}, i}\,(\rho(\vec{m}), l)\ \dfrac{\vdash \vec{q}, \vec{n}, j \vdash \vec{r}, l}{\vdash \vec{q}, \vec{r}, j}\,(\rho(\vec{n}), l)}{\dfrac{\vdash \vec{p}, \vec{q}, \vec{r}, \vec{r}, \rho(i,j)}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}}$$

The cases for (ext) are entirely similar to those for $(\otimes)$ except that we have the abstraction depth $\rho(i_1, i_2, j_1, j_2)$ in place of $\rho(i, j)$.

5.1.3 $(\&/\ ?\,cut)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{q}, \vec{n}, j}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \vec{\rho}(\vec{m}, \vec{n}), \rho(i,j)}\ \vdash \vec{r}, l}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \vec{r}, \rho(i,j)}\,(\rho(\vec{m}, \vec{n}), l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{r}, l}{\vdash \vec{p}, \vec{r}, i}\,(\rho(\vec{m}), l)\ \dfrac{\vdash \vec{q}, \vec{n}, j \vdash \vec{r}, l}{\vdash \vec{q}, \vec{r}, j}\,(\rho(\vec{n}), l)}{\vdash \vec{\rho}(\vec{p}, \vec{q}), \vec{r}, \rho(i,j)}$$

5.1.4 $(cut/\ ?\,cut1)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \vec{m}}\,(i,j)\ \vdash \vec{r}, l}{\vdash \vec{p}, \vec{q}, \vec{r}, \rho(i,j)}\,(\rho(\vec{m}), l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{r}, l}{\vdash \vec{p}, \vec{r}, i}\,(\rho(\vec{m}), l)\ \vdash \vec{q}, j}{\vdash \vec{p}, \vec{q}, \vec{r}}\,(i,j)$$

5.1.5 $(cut/\ ?\,cut2)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{q}, \vec{n}, j}{\vdash \vec{p}, \vec{q}, \vec{m}, \vec{n}}\,(i,j)\ \vdash \vec{r}, l}{\vdash \vec{p}, \vec{q}, \vec{r}}\,(\rho(\vec{m}, \vec{n}), l) \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \vdash \vec{r}, l}{\vdash \vec{p}, \vec{r}, i}\,(\rho(\vec{m}), l)\ \dfrac{\vdash \vec{q}, \vec{n}, j \vdash \vec{r}, l}{\vdash \vec{q}, \vec{r}, j}\,(\rho(\vec{n}), l)}{\dfrac{\vdash \vec{p}, \vec{q}, \vec{r}, \vec{r}}{\vdash \vec{p}, \vec{q}, \vec{r}}}\,(i,j)$$

The cases for $(\ ?\,cut)$ are entirely similar to those for (cut) except that we have $(\rho(\vec{i}), j)$–cut in place of (i, j)–cut.

5.2 Permutations with one-premiss rules

They are exactly similar to the permutations for (cut) except that we have to replace $\vec{k}$ and $\rho(\vec{k}, l)$ for k and (k, l), respectively. Hence we only give the computation of $(\,?\,cut/\,!\,)$–? which has not been covered yet.

5.2.1 $(\,?\,cut/\,!\,)$–?

$$\dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i}{\vdash \vec{p}, \vec{m}, i \quad \vdash \vec{q}, l}\,(\rho(\vec{m}), l)}{\vdash \vec{p}, \vec{q}, i} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \vec{p}, \vec{m}, i \quad \vdash \vec{q}, l}{\vdash \vec{p}, \vec{q}, i}\,(\rho(\vec{m}), l)}{\vdash \vec{p}, \vec{q}, i}$$

∎

Notice that the use of $(*/I)$ can indeed lead to the increase of the abstraction index as shown below:

$$(*/I) \quad \dfrac{\dfrac{\vdash \vec{p}, i \quad \vdash j, k}{\vdash \vec{p}, k}\,(i, j) \quad \vdash \vec{q}, l}{\vdash \vec{p}, \vec{q}}\,(k, l) \quad \rightsquigarrow \dfrac{\vdash \vec{p}, i \quad \vdash \vec{q}, l}{\vdash \vec{p}, \vec{q}}\,(i, l)$$

where $k < i$.

Proposition 13 *Suppose the proof π reduces to a proof τ without the use of $(*/I)$. Then the abstraction depth of each formula occurrence in the conclusion of π is less than or equal to the corresponding formula occurrence in τ, and the abstraction depth of π is less than or equal to of τ,* i.e., $\delta(\pi) \leq \delta(\tau)$.

Proof

When one of the reductions is applied to a subproof τ of a proof π, the abstraction indices of the cuts outside τ remain unchanged or decrease since the abstraction depth of each formula occurrence in the conclusion of π does not increase by the reduction. Given this, the result is immediate by induction on the number of one-step reductions. ∎

Proposition 14 *Consider the proof*

$$\pi \equiv \dfrac{\begin{array}{cc}\vdots\pi_1 & \vdots\pi_2\\ \vdash \Gamma, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}$$

and suppose that π contains only cuts over atomic formulas and the abstraction depth $\delta(\pi_1)$ and $\delta(\pi_2)$ are strictly less than the abstraction depth $\delta(\pi) = \delta(A) \times \delta(A^{\perp})$. Then there exists a proof τ such that

- $\pi \rightsquigarrow \tau$
- $\partial(\tau) \leq 1$

- $\delta(\tau) < \delta(\pi)$

Proof
Note that $1 \leq \delta(\pi)$. The proof is by induction on $\sigma(\pi)$.

Case 1 One of the cut formulas is not the main formula. Then apply one of the permutative reductions and use the inductive hypothesis.

Case 2 Otherwise we apply one of the symmetric reductions. This may create new cuts with formulas which are not atomic. However the abstraction indices of those newly created cuts are strictly less than $\delta(\pi)$ and they can be eliminated by Theorem 11 and Proposition 13 without increasing the abstraction depth of the proof.

∎

Lemma 15 *For any proof π with $1 \leq \delta(\pi)$ and $\partial(\pi) \leq 1$, there exists a proof τ such that*

- $\pi \rightsquigarrow \tau$
- $\partial(\tau) \leq 1$
- $\delta(\tau) < \delta(\pi)$

Proof
The proof is by induction on $\sigma(\pi)$. Consider the lower-most cut of the maximal abstraction index. We first apply the inductive hypothesis to both branches above the cut and eliminate the cut by Lemma 14. ∎

Theorem 16 *For any proof π, there exists a proof τ such that*

- $\pi \rightsquigarrow \tau$
- $\partial(\tau) \leq 1$
- $\delta(\tau) = 0$

Proof
First eliminate all the cuts over non-atomic formulas. Then the theorem follows from Lemma 15 by induction on $\delta(\pi)$. ∎

7 The elimination of atomic cuts involving axioms

In the previous two steps, we have reduced a proof π to a proof τ such that τ contains only atomic cuts one of whose cut formula is originated in an axiom. The last step of our cut-elimination is to eliminate all the remaining atomic cuts whose cut formulas are not Z-formulas. This is achieved by observing that the size of a proof decreases in the cut-elimination process when no contraction is used. For this, we need to make sure that the exponentials in Z-terms do not intervene in the cut-elimination.

Let's consider what kind of cuts still remain in the proof. First of all, we have cuts whose cut formulas are Z-formulas. As we have already remarked, we are not interested in eliminating those cuts. Secondly, we have cuts one of whose cut formulas is from the axioms other than (I). Those cuts can be eliminated easily without creating any new cuts. Thirdly, we have cuts both of whose cut formulas are originated in the axiom (I). They can be also eliminated without new cuts. Finally, we have the most interesting case in which the cut formulas are of the form $s \in t$ and $s \notin t$ and one of them is created by one of the abstraction rules and the other originated in the axiom (I).

In the last case, we have a cut

$$\frac{\begin{array}{c}\vdots\pi\\ \vdash s \in t, \Gamma\end{array} \quad \begin{array}{c}\\ \vdash s \notin t, s \in t\end{array}}{\vdash s \in t, \Gamma} \quad \text{or} \quad \frac{\begin{array}{c}\vdots\pi\\ \vdash s \notin t, \Gamma\end{array} \quad \begin{array}{c}\\ \vdash s \in t, s \notin t\end{array}}{\vdash s \notin t, \Gamma}$$

where $s \in t$ or $s \notin t$ on the left branch is created by one of the abstraction rules in the subproof π. We can assume that $s \in t$ and $s \notin t$ are not Z-formulas since otherwise we would not have to eliminate the cut. Furthermore, we know that the terms s and t are S-terms since otherwise $s \in t$ and $s \notin t$ would not be admissible. From this, we can conclude that t is not a Z-term as follows.

First, suppose that $s \in t$ is created by the abstraction rule $(\in Z)$ in the subproof π. Then s is a Z-term, and t has the form $\{v : A\}$. Among S-terms, only Z-terms have the form $\{v : A\}$, *i.e.*, abstraction over Z-variables. Hence t is a Z-term. Then both s and t are Z-terms, and we have already excluded this case from our consideration.

Therefore suppose that $s \in t$ is created by the rule $(\in S)$. Then t has the form $\{\mathbf{x} : A\}$ and hence it is not a Z-term. Similarly for the case where $s \notin t$ is created by one of the abstraction rules.

This observation ensures that we never encounter the exponetials in further unwinding the abstraction terms through the cut-elimination of the remaining atomic formulas.

Proposition 17 *All the permutative reductions for (cut) preserve the size of a proof.*

Proof
The proof is by inspection.

1 Permutative reductions with two-premiss rules

1.1 $(cut/\otimes)$, (cut/ext), (cut/cut) and $(cut/\ ?cut)$

$$\dfrac{\vdash \gamma \quad \dfrac{\vdash \alpha \quad \vdash \beta}{\vdash \alpha+\beta}}{\vdash \alpha+\beta+\gamma} \rightsquigarrow \dfrac{\dfrac{\vdash \gamma \quad \vdash \alpha}{\vdash \gamma+\alpha} \quad \vdash \beta}{\vdash \gamma+\alpha+\beta}$$

1.2 $(cut/\&)$

$$\dfrac{\vdash \gamma \quad \dfrac{\vdash \alpha \quad \vdash \beta}{\vdash \rho(\alpha,\beta)}}{\vdash \rho(\alpha,\beta)+\gamma} \rightsquigarrow \dfrac{\dfrac{\vdash \gamma \quad \vdash \alpha}{\vdash \alpha+\gamma} \quad \dfrac{\vdash \gamma \quad \vdash \beta}{\vdash \beta+\gamma}}{\rho(\alpha+\gamma,\beta+\gamma)=\rho(\alpha,\beta)+\gamma}$$

2 Permutative reductions with one-premiss rules

$$\dfrac{\vdash \gamma \quad \dfrac{\vdash \alpha}{\vdash \alpha+1}}{\vdash \alpha+\gamma+1} \rightsquigarrow \dfrac{\dfrac{\vdash \gamma \quad \vdash \alpha}{\vdash \alpha+\gamma}}{\vdash \alpha+\gamma+1}$$

∎

Proposition 18 *All the symmetric reductions for* (cut) *strictly decrease the size of a proof.*

Proof
The proof is by inspection.

1 $(\otimes/\parr)$

$$\dfrac{\dfrac{\vdash \alpha \quad \vdash \beta}{\vdash \alpha+\beta} \quad \dfrac{\vdash \gamma}{\vdash \gamma+1}}{\vdash \alpha+\beta+\gamma+1} \rightsquigarrow \dfrac{\vdash \beta \quad \dfrac{\vdash \alpha \quad \vdash \gamma}{\vdash \alpha+\gamma}}{\vdash \alpha+\beta+\gamma}$$

2 $(\&/\oplus 1)$ and $(\&/\oplus 2)$

$$\dfrac{\dfrac{\alpha \quad \beta}{\alpha+\beta} \quad \dfrac{\gamma}{\gamma+1}}{\alpha+\beta+\gamma+1} \rightsquigarrow \dfrac{\alpha \quad \gamma}{\alpha+\gamma}$$

3 $(\bigwedge Z/\bigvee Z)$, $(\bigwedge S/\bigvee S)$, $(\in Z/\notin Z)$, and $(\in S/\notin S)$

$$\dfrac{\dfrac{\vdash \alpha}{\vdash \alpha+1} \quad \dfrac{\vdash \beta}{\vdash \beta+1}}{\vdash \alpha+\beta+2} \rightsquigarrow \dfrac{\vdash \alpha \quad \vdash \beta}{\vdash \alpha+\beta}$$

4 $(\bot/\mathbf{1})$

$$\dfrac{\dfrac{\vdash \alpha}{\vdash \alpha+1} \quad \vdash 1}{\vdash \alpha+2} \rightsquigarrow \vdash \alpha$$

5 $(*/I)$

$$\dfrac{\vdash \alpha \quad \vdash 1}{\vdash \alpha+1} \rightsquigarrow \vdash \alpha$$

6 $(*/\top)$

$$\dfrac{\vdash \alpha \quad \vdash 1}{\vdash \alpha+1} \rightsquigarrow \vdash 1$$

∎

Lemma 19 *Suppose π contains only those cuts over S-formulas. Then there exists a proof τ such that*

- $\pi \rightsquigarrow \tau$
- $\sigma(\tau) \leq \sigma(\pi)$
- *τ contains only those cuts over atomic Z-formulas.*

Proof
The proof is by induction on $\sigma(\pi)$. If the last inference of π is (cut) with atomic Z-formulas or not (cut), then the conclusion follows immediately by the inductive hypothesis. Otherwise there are three cases.

Case 1 One of the permutative reductions except (cut/cut) is applicable to the cut at the bottom. Then we perform the permutative reduction and then apply the inductive hypothesis.

Case 2 The permutation (cut/cut) is applicable. If the upper cut is with an atomic Z-formula, apply the permutation and use the inductive hypothesis. Otherwise apply the inductive hypothesis to the upper cut. This reduction does not increase the size of the proof and leads to the situation to which the other cases are applicable.

Case 3 One of the symmetric reductions is applicable. Then we perform the reduction. By construction of S-formulas, the cut formulas of the newly created cuts are again S-formulas. Since the reduction strictly decreases the size of a proof, we can apply the inductive hypothesis.

∎

Theorem 20 *For any proof π, there exists a proof τ such that $\pi \rightsquigarrow \tau$ and τ contains only those cuts over atomic Z-formulas.*

Proof
By Theorem 16, π reduces to a proof whose only cut formulas are atomic formulas. Furthermore, at least one of the cut formulas for a remaining cut is originated in an axiom. If one of them is created by one of the abstraction rules and the other is from the axiom (I), the cut-formulas must be S-formulas. For other cases, we can apply permutative and symmetric reductions to remove cuts whose cut-formulas are not S-formulas. The proof thus obtained satisfies the condition of Lemma 19. ∎

8 The consistency of LZF

Given all the previous results at hand, we can now prove that LZF is a conservative extension of ZF^-. For this purpose, we consider ZF^- formulated as one-sided sequent calculus with the abstraction terms introduced in Section 2.1. We then use the translation $(\ )^+$ extended by induction on the construction of terms and formulas in a natural way:

- *The translation of a variable v is v itself regarded as a Z-variable;*
- *The translation of an abstraction term $\{v : A\}$ is $\{v : A^+\}$ where the abstraction is construed as over a Z-variable;*
- $(s \in t)^+ \equiv ?\,!(s^+ \in t^+)$ *and similarly for $s = t$;*
- *The translation of complex formulas are as before.*

When Γ is the multiset of formulas $A_1, \ldots, A_k$, we write Γ^+ to denote the multiset of ${A_1}^+, \ldots, {A_k}^+$.

Proposition 21 *Let A be a formula of ZF^-. Then, there exists a formula C of LZF such that A^+ is logically equivalent to $?\,!C$.*

Proof
The proof is by induction on the construction of terms and formulas of ZF^-. For the translations of formulas $D \vee E$ and $\exists x D$, we use the inductive hypotheses and show that $?\,!D_1 \parr ?\,!E_1$ and $?\bigvee u.\,!?\,!D_1$ are logically equivalent to $?\,!(?\,!D_1 \parr ?\,!E_1)$ and $?\,!\bigvee u.\,?\,!D_1$, respectively. Other cases are straightforward. ∎

Theorem 22 *Suppose that the sequent* $\vdash \Gamma$ *is provable in ZF*$^-$*. Then the sequent* $\vdash \Gamma^+$ *is also provable in LZF.*

Proof
The proof is by induction on the proof construction of ZF$^-$. For most of the cases, we need to introduce additional exponentials and the use of $(!)$ is justified due to Proposition 21. For the translations of axioms, we have

$$\frac{\dfrac{\vdash A, A^{\perp}}{\vdash A, ?\,A^{\perp}}}{\vdash ?\,!\,A, ?\,!\,?\,A^{\perp}} \qquad \frac{\dfrac{\vdash s \neq t, A, A[t/s]^{\perp}}{\vdash ?\,s \neq t, A, ?\,A[t/s]^{\perp}}}{\vdash ?\,!\,?\,s \neq t, ?\,!\,A, ?\,!\,?\,A[t/s]^{\perp}}$$

where we combine the successive applications of $(!)$ and $(D\ ?)$ as one step.

For $(\wedge)$ and (cut),

$$\frac{\dfrac{\vdash \Gamma^+, A^+}{\vdash \Gamma^+, !\,A^+} \quad \dfrac{\vdash \Delta^+, B^+}{\vdash \Delta^+, !\,B^+}}{\dfrac{\vdash \Gamma^+, \Delta^+, !\,A \otimes !\,B,}{\vdash \Gamma^+, \Delta^+, ?\,(!\,A \otimes !\,B)}} \qquad \frac{\dfrac{\vdash \Gamma^+, A^+}{\vdash \Gamma^+, !\,A^+} \quad \vdash \Delta^+, ?\,A^{+\perp}}{\vdash \Gamma^+, \Delta^+}$$

For the abstraction rules, we first note that $!\,?\,C$ is logically equivalent to $!\,?\,!\,?\,C$. Hence, $A^{+\perp}$ is logically equivalent to $!\,?\,A^{+\perp}$ due to Proposition 21. Then, we have

$$\frac{\dfrac{\vdash \Gamma^+, A^+[s^+/u]}{\vdash \Gamma^+, s^+ \in \{u : A^+\}}}{\vdash \Gamma^+, ?\,!\,(s^+ \in \{u : A^+\})} \qquad \frac{\dfrac{\dfrac{\dfrac{\vdash \Gamma^+, ?\,A^+[s^+/u]^{\perp}}{\vdash \Gamma^+, !\,?\,A^+[s^+/u]^{\perp}}}{\vdash \Gamma^+, A^+[s^+/u]^{\perp}}}{\vdash \Gamma^+, s^+ \notin \{u : A^+\}}}{\dfrac{\vdash \Gamma^+, ?\,(s^+ \notin \{u : A^+\})}{\vdash \Gamma^+, ?\,!\,?\,(s^+ \notin \{u : A^+\})}}$$

where the inference at the double bar is justfied by the logical equivalence.

For the extensionality, note that $?\,!\,A$ implies $?\,A$, and $?\,!\,?\,A$ implies $?\,A$ as well. Hence, we have

$$\frac{\dfrac{\dfrac{\vdash ?\,!\,(u \in s^+), ?\,!\,?\,(u \notin t^+), \Gamma^+}{\vdash ?\,(u \in s^+), ?\,(u \notin t^+), \Gamma^+} \quad \dfrac{\vdash ?\,!\,?\,(v \notin s^+), ?\,!\,(v \in t^+), \Delta^+}{\vdash ?\,(v \notin s^+), ?\,(v \in t^+), \Delta^+}}{\vdash s^+ = t^+, \Gamma^+, \Delta^+}}{\vdash ?\,!\,(s^+ = t^+), \Gamma^+, \Delta^+}$$

where the inference at the double bar is justified by the above implications. Other cases are straightforward. ∎

Theorem 23 *Let* A^+ *be the translation of a formula* A *in ZF*$^-$*. If* A^+ *is provable in LZF, then so is* A *in ZF*$^-$*.*

Proof
Suppose that there is a proof π of $\vdash A^+$ in LZF. Then there is a proof of the same sequent which only contains Z-terms. Ignoring all the occurrences of ! and ?, we can read the proof τ as a proof in ZF^-. ∎

Theorem 24 *LZF is consistent relative to ZF^-.*

Proof
If A and $A^\perp$ are both provable in LZF, then any formula of the form $?C$ is provable. In particular, $?!?(u \neq u)$ is provable in LZF and hence $\neg u = u$ is provable in ZF^-. This contradicts the consistency of ZF^-. ∎

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