An $O(nL)$ Iteration Algorithm for Computing Bounds in Quadratic Optimization Problems

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Abstract

We consider the problem of optimizing a quadratic function subject to integer constraints. This problem is NP-hard in the general case. We present a new polynomial time algorithm for computing bounds on the solutions to such optimization problems. We transform the problem into a problem for minimizing the trace of a matrix subject to positive definiteness condition. We then propose an interior-point method to solve this problem. We show that the algorithm takes no more than $O(nL)$ iterations (where L is the the number of bits required to represent the input). The algorithm does two matrix inversions in each iteration .

Keywords: Bounds, complexity, quadratic optimization, interior point methods.

1 Outline

The second section of the paper shall introduce the problem of computing upper bounds on a quadratic optimization problem. We shall also motivate an interior point approach to solving the problem. The third section gives an interior point method for solving the problem. The algorithm described in the third section assumes knowledge of the optimal value of the objective function. In the fourth section, we analyse the complexity of this algorithm. Then in the fifth section, we shall generalize the algorithm to the case where we do not have knowledge about the optimal value. Finally, the conclusions are presented in the sixth section.

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2 Introduction

Consider the quadratic optimization problem

$$\min f(x) = x^T Q x$$

$$s.t.\ x \in S = \{-1,1\}^n \tag{1}$$

where $Q \in R^{n\times n}$ is a real symmetric matrix. Let f_{min} be the optimal solution to problem (1).

In this paper we shall be addressing the problem of finding good lower bounds on f_{min}. Note that if problem (1) were solvable in polynomial time we could obtain a lower bound by simply computing the value of f_{min}. However problem (1) is NP-hard [2] in the general case and hence computing f_{min} may be difficult. The problem of computing the lower bound needs to be contrasted with that of finding the upper bound which is comparatively easy since we simply need to compute $f(x)$ at some x in S.

To apply an interior point method to this problem we need to embed the discrete set S in a continuous set $T \supseteq S$. Notice that the minimum of $f(x)$ over T is a lower bound on f_{min}.

A commonly used approach is to choose the continuous set to be the box B defined as follows

$$B = \{x \in R^n | -1 \le x_i \le 1, i = 1,\ldots,n\}.$$

The quadratic optimization problem on a box can be solved in polynomial time if Q is a positive definite matrix [3] [9] [10] [11]. But if Q doesn't satisfy positive definiteness then the problem is again shown to be NP-hard [13] [12]. We are interested in solving the difficult problem in which Q has at least one negative eigenvalue [1].

We observed that optimizing over a box can be hard and so instead we choose to enclose the box in an ellipsoid E that contains the box B. Let

$$U = \{w = (w_1,\ldots,w_n) \in R^n | \sum_{i=1}^{n} w_i = 1 \ and \ w_i > 0\}.$$

Consider the ellipsoid

$$E(w) = \{x \in R^n | x^T W x \le 1\}$$

where $w \in U$ and $W = diag(w)$.

The set S is contained in $E(w)$. Let $\lambda_{min}(w)$ be the minimum eigenvalue of $W^{-1/2}QW^{-1/2}$ then we have

$$\min \frac{x^T Q x}{x^T W x} = \min \frac{x^T W^{-1/2} Q W^{-1/2} x}{x^T x} = \lambda_{min}(w),$$

and so we may conclude that

$$x^T Q x \ge \lambda_{min}(w)\,,\ \forall x \in E(w).$$

Hence the minimum value of $f(x)$ over $E(w)$ can be obtained by simply computing the minimum eigenvalue of $W^{-1/2}QW^{-1/2}$. To obtain a better bound on f_{min} we need to optimize $\lambda_{min}(w)$ over the set U.

3 Interior Point Approach to the Problem

We have transformed the problem of finding an upper bound into the following optimization problem

$$\max \mu$$

subject to

$$\frac{x^T Q x}{x^T W x} \geq \mu\,,\ \forall x \in R^n - \{0\}\ and\ w \in U.$$

We can further simplify the problem by defining new variables $d = (d_1, \ldots, d_n) \in R^n$ where $\sum_{i=1}^n d_i = 0$. Let $D = diag(d)$ then we notice that if

$$\frac{x^T (Q - D) x}{x^T W x} \geq \mu$$

then since $\sum_{i=1}^n w_i = 1$ and $\sum_{i=1}^n d_i = 0$ we may conclude that for $x \in S$ we have

$$x^T Q x \geq \mu x^T W x + x^T D x = \mu.$$

Let $z = \mu w + d$ and $Z = diag(z)$ then for all $x \in S$ we get

$$x^T Z x = e^T z = \mu$$

and so the problem transforms into

$$\max e^T z$$

$$s.t.\ x^T (Q - Z) x \geq 0.$$

Let $M(z) = Q - Z$. We observe that solving above problem amounts to minimizing the trace of $M(z)$ while keeping $M(z)$ positive semi-definite. Let $\lambda_i(M(z))$ denote the i^{th} eigenvalue of $M(z)$ [14]. Since $M(z)$ is a real and symmetric matrix, it has n real eigenvalues $\lambda_i(M(z)), i = 1, \ldots, n$. To ensure positive definiteness, the eigenvalues of $M(z)$ must be nonnegative. Hence the above problem may be reformulated as follows

$$\min tr(M(z))$$

$$s.t.\ \lambda_i(M(z)) \geq 0\,,\ i = 1, \ldots, n \tag{2}$$

3.1 Centering Transformation

We shall be using an iterative interior point approach to solve problem (2). As in linear programming we would like to define a centering transform that centers the current iterate with respect to the region over which we are optimizing. Hence we examine the following family of transformations.

Let $P = \{A \in R^{n \times n} | A = A^T \text{ and } x^T A x > 0, \forall x \in R^n - \{0\}\}$ be a set of positive definite matrices. We define a family G of transformation $L_F : P \to P$ where F is an $n \times n$ real and invertible matrix. The transformations are defined as

$$L_F(A) = FAF^T.$$

The transformations are one-one and onto and they form a Lie group under the composition operation.

3.2 Properties of Transformation

In this subsection, we shall prove some lemmas about the transformations in G and about optimization over the transformed space.

Lemma 3.1 Centering Property
Given any positive definite matrix M there exists $L_F \in G$ such that $L_F(M) = I$.

Proof:
Since M is a positive definite matrix, we know that $M^{-1/2}$ exists.
Let $F = M^{-1/2}$ then note that $L_F(M) = I$. ∎

If we are at a point z^o then $L_{M(z^o)^{-1/2}}$ maps the matrix $M(z^o)$ to the identity matrix which has all eigenvalues equal to 1. Hence it can be used as a centering transform in our interior point method for solving problem (2).

To measure distances in the transformed space, we define a Riemannian metric [6][5]. Let $F = M(z^o)^{-1/2}$. The Riemannian distance between the points z^o and z that is invariant under the transformation L_F is denoted by $d^2(M(z), M(z^o))$.

Lemma 3.2 Riemannian metric in transformed space
Let $z, z^o \in R^n$ be any two points in the interior of the region of optimization and let the matrix $H(z^o)$ be defined as

$$[H(z^o)]_{ij} = (e_i^T M(z^o)^{-1} e_j)^2$$

then the distance in the Riemannian metric may be expressed as

$$d^2(M(z), M(z^o)) = (z - z^o)^T H(z^o)(z - z^o).$$

Proof:
Let $\Delta z = z - z^o$ and $\Delta Z = diag(\Delta z)$ then we have

$$L_F(M(z)) = I + \Delta M'$$

where $\Delta M' = F\Delta Z F$.

Since the Riemannian distances are invariant under transform L_F we have

$$\begin{aligned}
d^2(M(z), M(z^o)) &= d^2(L_F(M(z)), L_F(M(z^o))) \\
&= d^2(I, I + \Delta M') \\
&= tr(\Delta M' \Delta M'^T) \\
&= tr(F\Delta Z F \Delta Z F) \\
&= \sum_i \sum_j (e_i^T M(z^o)^{-1} e_j)^2 \Delta z_i \Delta z_j \\
&= (z - z^o)^T H(z^o)(z - z^o).
\end{aligned}$$

∎

The following lemma will describe an inscribed ball in the feasible region for problem (2).

Lemma 3.3 Inscribed Open Ball in the transformed space
An open ball of radius 1 centered at z^o in the transformed space is contained in the interior of the feasible region for problem (2).

Proof:
If z is within an open ball of radius 1 centered at z^o in the transformed space then

$$d^2(M(z), M(z^o)) < 1$$

and hence we get

$$tr(M(z^o)^{-1/2} \Delta Z M(z^o)^{-1} \Delta Z M(z^o)^{-1/2}) < 1.$$

In other words we can say that

$$\sum_{i=1}^{n} \lambda_i^2 (M(z^o)^{-1/2} \Delta Z M(z^o)^{-1/2}) < 1.$$

But we know that

$$\lambda_i(L_F(M(z))) = 1 + \lambda_i(M(z^o)^{-1/2} \Delta Z M(z^o)^{-1/2})$$

and hence we can conclude that

$$\lambda_i(L_F(M(z))) > 0.$$

Consequently, we get

$$\lambda_i(M(z)) > 0.$$

Hence z is an interior point of (2). ∎

In the next lemma, we show how we can optimize a linear function on an open ball in the transformed space.

Lemma 3.4 Optimizing Linear function over a ball
Consider the following optimization problem

$$\min c^T z$$

$$s.t.\ d^2(M(z), M(z^o)) \leq \alpha^2. \tag{3}$$

The optimal solution to problem (3) is given by

$$z = z^o - \beta H(z^o)^{-1} c$$

where $\beta = \frac{\alpha}{\sqrt{c^T H(z^o)^{-1} c}}$.

Proof:
We notice that problem (3) corresponds to minimizing $c^T z$ over a ball of radius α in the transformed space. Hence by lemma (2) the problem may be rewritten as

$$\min c^T z$$

$$s.t. (z - z^o)^T H(z^o)(z - z^o) \leq \alpha^2$$

where $[H(z^o)]_{ij} = (e_i^T M(z^o)^{-1} e_j)^2$.

The matrix $H(z^o)$ can be shown to be positive definite. Hence by the Kuhn-Tucker conditions [8], the optimal solution to the above problem (3) can be obtained by solving the linear system given by

$$H(z^o)(z - z^o) = -\beta c$$

where $\beta = \frac{\alpha}{\sqrt{c^T H(z^o)^{-1} c}}$. ∎

We now consider a potential function corresponding to the constraints in (2). The potential function is defined as

$$\phi(z) = -\sum_{i=1}^{n} \ln \lambda_i(M(z)) = -\ln \det(M(z)).$$

The potential function is defined only over the interior of the region of optimization in problem 2.

Lemma 3.5 Invariance of Potential Function
The potential function $\phi(z)$ is invariant under the transformations in G.

Proof:
We notice that

$$\begin{aligned}\phi(L_F(A)) - \phi(L_F(B)) &= -\ln\det(FAF^T) - (-\ln\det(FBF^T)) \\ &= -\ln\frac{\det(F)\det(A)\det(F)}{\det(F)\det(B)\det(F)} \\ &= \phi(A) - \phi(B).\end{aligned}$$

∎

In the next subsection, we shall use the properties that we proved in the above lemmas, to construct an interior point method to solve the optimization problem (2).

3.3 Description of an Algorithm

Our interior point method to solve problem (2) is an iterative method. We shall first describe a simple method that assumes knowledge of the optimal value v^* of the objective function in (2). Later we shall modify the algorithm to work in the case where the value of v^* is not known.

The method uses the potential function defined as

$$f(z) = q\ln(tr((M(z)) - v^*)) - \sum_{i=1}^{n}\ln\lambda_i(M(z)) \tag{4}$$

where $q = 2n$.

We start with some value $z^{(0)}$ in the interior of the region of optimization. At iteration K, let $z^{(K)}$ denote the current value of the iterate. The algorithm may then be described by the following sequence of steps:

1. start with some feasible interior point $z^{(0)}$.
2. Apply the centering transform $L_{M(z^{(K)})^{-1/2}}$.
3. Construct a linear approximation to the potential function $f(z)$ at $z^{(K)}$.
4. find a point $z^{(K+1)}$ that optimizes this linear approximation over a ball of radius $\alpha < 1$ centered at $z^{(K)}$ in the transformed space.
5. Move to the new point $z^{(K+1)}$.
6. Repeat steps 2-6 until we are reasonably close to the solution.

Note that a ball of radius α will be completely contained in the feasible region by lemma 3 and hence the point $z^{(K+1)}$ is strictly in the interior of the feasible region. Also, the optimization over the ball may be done by simply solving a linear system of equations as shown in lemma 4.

4 Complexity Analysis

We shall measure the progress made by our algorithm in terms of the reduction we obtain in the potential function defined in (4). Let L_F be the centering transform and let $\tilde{M}(z) = FM(z)F$ then the transformed potential function is given by

$$\tilde{f}(z) = q \ln(tr(M(z)) - v^*) - \sum_{i=1}^{n} \ln \lambda_i(\tilde{M}(z)). \tag{5}$$

Note that changes in the potential function are invariant with respect to the transformation and we can just as well work with the transformed potential function (5). The linear approximation to (5) at the point $z^{(K)}$ is given by

$$\tilde{f}_1(z) = \tilde{f}(z^{(K)}) + q\frac{tr(M(z)) - tr(M(z^{(K)}))}{tr(M(z^{(K)})) - v^*} - (tr(\tilde{M}(z)) - tr(\tilde{M}(z^{(K)}))). \tag{6}$$

At each iteration we shall seek to reduce the potential function by minimizing its linear approximation.

4.1 Reduction of Potential function

We shall first show that the linear approximation (6) to the potential function can be reduced by a constant.

Lemma 4.1 *:*
Let $z^{(K+1)}$ be the point that minimizes $\tilde{f}_1(z)$ on a ball of radius α centered at $z^{(K)}$ in the transformed space at iteration K. Let $\Delta\tilde{f}_1^{(K)} = \tilde{f}_1(z^{(K+1)}) - \tilde{f}_1(z^{(K)})$ then

$$\Delta\tilde{f}_1^{(K)} \leq -\alpha.$$

Proof:
Let z^* be an optimal solution to problem (2). Let the Riemannian distance between z^* and $z^{(K)}$ be R i.e. we have

$$(z^* - z^{(K)})^T H(z^{(K)})(z^* - z^{(K)}) = \sum_{i=1}^{n}(\lambda_i(\tilde{M}(z^*)) - 1)^2 = R^2.$$

If we join the current iterate $z^{(K)}$ to the point z^* , it will intersect the ball of radius α in the transformed space at a point $z^\alpha = (1-\lambda)z^{(K)} + \lambda z^*$ where $\lambda = \alpha/R$. The change in $\tilde{f}_1(z)$ at the point z^α is given by

$$\begin{aligned}
\Delta\tilde{f}_1^\alpha &= \tilde{f}_1(z^\alpha) - \tilde{f}_1(z^{(K)}) \\
&= q\frac{tr(M((1-\lambda)z^{(K)} + \lambda z^*)) - tr(M(z^{(K)}))}{tr(M(z^{(K)})) - tr(M(z^*))} \\
&\quad -(tr(\tilde{M}((1-\lambda)z^{(K)} + \lambda z^*)) - tr(\tilde{M}(z^{(K)}))) \\
&= -q\lambda - \lambda tr(\tilde{M}(z^*) - I).
\end{aligned}$$

Let $\Gamma = \{i | \lambda_i(\tilde{M}(z^*)) < 1\}$. We know that $\lambda_i(\tilde{M}(z^*)) \geq 0$ and hence we get

$$\begin{aligned} tr(\tilde{M}(z^*) - I) &= \sum_{i \in \Gamma} (\lambda_i(\tilde{M}(z^*)) - 1) + \sum_{i \notin \Gamma} (\lambda_i(\tilde{M}(z^*)) - 1) \\ &\geq -n + \begin{cases} 0 & if\ R^2 < n \\ \sqrt{R^2 - n} & if\ R^2 \geq n \end{cases}. \end{aligned}$$

Hence we conclude that the change in the linear approximation to the potential function is bounded by

$$\Delta \tilde{f}_1^{\alpha} \leq -n\lambda - \lambda \begin{cases} 0 & if\ R^2 < n \\ \sqrt{R^2 - n} & if\ R^2 \geq n \end{cases}.$$

If $\lambda \geq \alpha/n$ then clearly $\Delta \tilde{f}_1^{\alpha} \leq -\alpha$. On the other hand, if $\lambda < \alpha/n$ then since $\lambda R = \alpha$, we can again show that $\Delta \tilde{f}_1^{\alpha} \leq -\alpha$. Since $z^{(K+1)}$ minimizes $\tilde{f}_1(z)$ on the ball of radius α

$$\Delta \tilde{f}_1^{(K)} \leq \Delta \tilde{f}_1^{\alpha} \leq -\alpha.$$

∎

So we have shown that the linear approximation of $\tilde{f}(z)$ can be reduced by a constant. To determine the corresponding reduction in the potential function we need to determine how close the potential function value is to its linear approximation.

Lemma 4.2 *:*
Let $z^{(K+1)}$ be the point that minimizes $\tilde{f}_1(z)$ on the ball of radius α centered at $z^{(K)}$ in the transformed space at iteration K. Then the reduction in the potential function at iteration K is bounded by

$$\tilde{f}(z^{(K+1)}) - \tilde{f}(z^{(K)}) \leq -\alpha + \frac{\alpha^2}{2(1-\alpha)}.$$

Proof:
Consider the two terms in the potential function given by $\Psi(z) = q \ln(tr(M(z)) - v^*)$ and $\Phi(z) = -\sum_{i=1}^{n} \ln(\lambda_i(\tilde{M}(z)))$. Let $\Psi_1(z)$ and $\Phi_1(z)$ denote their respective linear approximations. Now since $\Psi(z)$ is a concave function we have $\Psi(z) \leq \Psi_1(z)$. On the other hand, since $\ln(1-\lambda) \geq \lambda - \frac{\lambda^2}{2(1-|\lambda|)}$ for $|\lambda| < 1$ we get the relation that

$$\Phi(z^{(K+1)}) \leq \Phi_1(z^{(K+1)}) + \frac{1}{2(1-\alpha)} \sum_{i=1}^{n} (\lambda_i(\tilde{M}(z)) - 1)^2 \leq \Phi_1(z^{(K+1)}) + \frac{\alpha^2}{2(1-\alpha)}.$$

Hence the reduction in the potential function is bounded by

$$\tilde{f}(z^{(K+1)}) - \tilde{f}(z^{(K)}) \leq \Delta \tilde{f}_1^{(K)} + \frac{\alpha^2}{2(1-\alpha)} \leq -\alpha + \frac{\alpha^2}{2(1-\alpha)}.$$

∎

Let $\delta = -\alpha + \frac{\alpha^2}{2(1-\alpha)}$ then for $\alpha = 1/2$ we get $\delta = -1/4$. Thus we can reduce the potential function by a constant at each iteration.

4.2 Convergence to the optimal solution

Let λ_{min} be the minimum eigenvalue of Q then if we initialize $z^{(0)} = (\lambda_{min} - 1)e$ where $e = (1, \dots, 1) \in R^n$ then we know that $Q - Z^{(0)}$ is a positive definite matrix and hence $z^{(0)}$ is in the interior of the region of optimization.

Lemma 4.3 *:*
If λ_{max} is the maximum absolute eigenvalue of Q and we need the optimal value of the objective to be within a precision of ϵ then the algorithm terminates in $O(n \ln(\frac{2n\lambda_{max}+n}{\epsilon}))$ iterations.

Proof:
Since $Q - Z^*$ is a positive definite matrix we know that $v^* \geq 0$. Hence we have

$$f^{(0)}(z^{(0)}) \leq q \ln tr(M(z^{(0)})) - \ln det(M(z^{(0)})).$$

Let N be the number of iterations required to attain the optimal solution within a precision of ϵ then

$$f(z^{(N)}) \leq f(z^{(0)}) - N\delta.$$

This gives us the following inequalities.

$$\begin{aligned} N\delta &\leq q \ln(\frac{tr(M(z^{(0)}))}{\epsilon}) + \ln(\frac{det(M(z^{(N)}))}{det(M(z^{(0)}))}) \\ &\leq q \ln(\frac{2n\lambda_{max} + n}{\epsilon}) + n \ln(\frac{2n\lambda_{max} + n}{n}). \end{aligned}$$

Hence we have

$$N \leq \frac{3n}{\delta} \ln(\frac{2n\lambda_{max} + n}{\epsilon}).$$

∎

If the matrix Q is over integers then $\epsilon = 1$ should be sufficient. If the number of bits in the input is L then we can in fact show that $O(nL)$ iterations are sufficient to obtain the solution. Each iteration requires two matrix inversions . If we use fast matrix multiplication to do matrix inversion then the worst case complexity of the algorithm is $O(n^{3.38}L)$.

5 Generalizing the Algorithm

The algorithm in the previous section, assumed knowledge about the optimal value v^* of the objective in problem (2). But in practice this value may not be known. Hence the potential function $f(z)$ is not known. In this section we show how the previous algorithm can be adapted to make do without the value of v^*.

Let us consider the parametric family of potential functions given by

$$g(z,v) = q\ln(tr(M(z)) - v) - \ln det(M(z)).$$

where $v \in R$ is a parameter.

Our new algorithm will generate a monotonically increasing sequence of parameters $v^{(K)}$ that converges to the optimal value v^*. We shall in the process get a sequence of functions $g(z, v^{(K)})$ that converges to the desired potential function $g(z, v^*) = f(z)$. Note that since $v^{(K)} \leq v^*$ at each iteration $g(z, v^{(K)}) \geq f(z)$ for all z in the domain of f.

The sequence $v^{(K)}$ is constructed in conjunction with the sequence $z^{(K)}$ of interior points. The algorithm to do this outlined in Figure 1. Since $Q - Z^*$ is a positive definite matrix we may use $v^{(0)} = 0 \leq v^*$ as the starting point in the sequence. For this method to terminate in a polynomial number of iterations, we must give a technique for updating $v^{(K)}$ at each iteration, so that the potential function is reduced substantially at each iteration.

Let $g_1^{(K)}(z,v)$ denote the linear approximation of $g(z,v)$ at $z^{(K)}$. Let $\sigma(v) = \frac{q}{tr(M(z^{(K)}) - v}$. Then we can write

$$g_1^{(K)}(z,v) = -\sigma(v)e^T z + \nabla\Phi(z^{(K)})^T z + const.$$

We shall show how $g_1^{(K)}(z,v)$ can be reduced by a constant amount at each iteration.

Lemma 5.1 *:*
We can find $v^{(K+1)} \in R$ and a point $z^{(K+1)}$ in a closed ball of radius α centered at $z^{(K)}$ such that $v^{(K)} \leq v^{(K+1)} \leq v^*$ and

$$g_1^{(K)}(z^{(K+1)}, v^{(K+1)}) - g_1^{(K)}(z^{(K)}, v^{(K+1)}) \leq -\alpha.$$

Proof:
Let $\Delta g_1^\alpha(v)$ be a function in v that denotes the amount by which $g_1^{(K)}(z,v)$ can be decreased over a ball of radius α centered at $z^{(K)}$. Then by Lemma 3.4 we may conclude that

$$\Delta g_1^\alpha(v) = -\alpha\sqrt{(-\sigma(v)e + \nabla\Phi(z^{(K)}))^T H(z^{(K)})(-\sigma(v)e + \nabla\Phi(z^{(K)}))}. \tag{7}$$

Now if can find $v^{(K+1)} \leq v^*$ such that $\Delta g_1^\alpha(v^{(K+1)}) \leq -\alpha$ then we can find the corresponding $z^{(K+1)}$ by simply minimizing $g_1^{(K)}(z, v^{(K+1)})$ over a ball of radius α centered at $z^{(K)}$.

We need to consider the following 2 cases:

Case 1: $\Delta g_1^\alpha(v^{(K)}) \leq -\alpha$
In this case we simply set $v^{(K+1)} = v^{(K)}$.

Case 2: $\Delta g_1^\alpha(v^{(K)}) > -\alpha$
In this case we need to increase the value of v. We know from Lemma 4.1 that

$g_1^\alpha(v^*) = \Delta \tilde{f}_1^{(K)} \leq -\alpha$. Moreover, we know that $\Delta g_1^\alpha(v^{(K)}) > -\alpha$. But $\Delta g_1^\alpha(v)$ is a continuous function in v over the closed interval $[v^{(K)}, v^*]$. Hence by the Intermediate Value Theorem there exists a point $v^{(K+1)} \in [v^{(K)}, v^*]$ such that $\Delta g_1^\alpha(v^{(K+1)}) = -\alpha$. To compute the exact value of $v^{(K+1)}$ we need to simply solve a quadratic equation in v arising from (7) to get a value of $v \in [v^{(K)}, v^*]$ that gives the necessary reduction. This proves the lemma for the second case.

∎

The above lemma gives us a technique for updating $v^{(K)}$ and $z^{(K)}$ at each iteration. To prove convergence of this method, we prove the following lemma.

Lemma 5.2 *:*
Let $z^{(K)}$ be the current interior point and $v^{(K)} \leq v^*$ be the current estimate of the optimal value. Let $z^{(K+1)}$ and $v^{(K+1)}$ be the new interior point and new estimate respectively, that are obtained by applying Lemma 5.1 then

$$g(z^{K+1}, v^{(K+1)}) - g(z^K, v^{(K)}) \leq -\alpha + \frac{\alpha^2}{2(1-\alpha)}.$$

Proof:
We know from Lemma 5.1 that

$$g_1^{(K)}(z^{(K+1)}, v^{(K+1)}) - g_1^{(K)}(z^{(K)}, v^{(K+1)}) \leq -\alpha.$$

Now using an argument similar to that in Lemma 4.2 we can show that

$$g(z^{(K+1)}, v^{(K+1)}) - g(z^{(K)}, v^{(K+1)}) \leq -\alpha + \frac{\alpha^2}{2(1-\alpha)}.$$

But since $v^{(K)} \leq v^{(K+1)}$ we know that

$$g(z^{(K)}, v^{(K+1)}) \leq g(z^{(K)}, v^{(K)}).$$

and hence the result follows. ∎

Now we can prove the convergence result for our generalized algorithm.

Lemma 5.3 *:*
The algorithm described in Figure 1 terminates in $O(nL)$ iterations.

Proof:
From Lemma 5.2, we know that at each iteration, we reduce the parametrized potential function $g(z, v)$ by at least a constant amount $\delta = -\alpha + \frac{\alpha^2}{2(1-\alpha)}$. If the algorithm takes N iterations to converge then we have

$$f(z^{(N)}) \leq g(z^{(N)}, v^{(N)}) \leq g(z^{(0)}, v^{(0)}) - N\delta.$$

We can then use an analysis similar to that in Lemma 4.3 to prove a similar bound on the number of iterations. ∎

Lower bound*(in: Q, ϵ, out: z, opt)*

begin

1. $z^{(0)} = (\lambda_{min}(Q) - 1)e \ ; \ v^{(o)} = 0$
2. $M(z^{(0)}) = Q - Z^{(0)}$; $K = 0$

   While $tr(M(z^{(K)})) - v^{(K)}) \geq \epsilon$ do

3. Construct $H^{(K)}$ where $H_{ij}^{(K)} = (e_i^T M(z^{(K)})^{-1} e_j)^2$
4. Define $f^{(K)}(z) = q \ln(tr(M(z^{(K)})) - v^{(K)}) - \ln det M(z^{(K)})$
5. Find $g^{(K)} = \nabla f^{(K)}(z^{(K)})$
6. Solve linear system
   $$H^{(K)} \Delta z = -\beta g^{(K)}$$
   where $\beta = \frac{0.5}{\sqrt{g^{(K)T} H^{(K)-1} g^{(K)}}}$
7. If $g^{(K)T} \Delta z < 0.5$ then increase $v^{(K)}$ until $g^{(K)T} \Delta z = 0.5$
8. $z^{(K+1)} = z^{(K)} + \Delta z \ ; \ K = K + 1$

   endwhile

9. $z = z^{(K)}, \ opt = tr(Q) - v^{(K)}$

end

Figure 1: Algorithm for computing Lower Bounds

6 Conclusions

We have described an $O(n^{3.38}L)$ algorithm for finding a lower bound on quadratic minimization problems subject to integer constraints. We have analyzed the complexity of the algorithm in this paper and are working on further reducing this complexity. We also hope to extend the techniques developed here to solve other optimization problems. A more generalized version of our problem can be found in [7]. The analysis of the quality of the lower bound that can be obtained by this method is an interesting open problem.

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