TRANSITIVE FORK ENVIRONMENTS AND MINIMUM COST MULTIFLOWS

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June 1993

Abstract. The following minimum cost maximum multiflow problem is the focus of the paper: $(*)$ given an undirected graph $G = (VG, EG)$, a subset $T \in VG$ (of terminals), and functions $c : EG \to \mathbb{Z}_+$ (of capacities) and $a : EG \to \mathbb{Z}_+$ (of costs), find a collection f of flows (a *multiflow*) connecting arbitrary pairs of distinct terminals so that the total flow $\zeta^f(e)$ through each edge does not exceed its capacity $c(e)$, and: (a) the sum of values of partial flows is maximum; and (b) the total cost $\sum_{e \in EG} a(e)\zeta^f(e)$ of f is as small as possible, subject to (a). For $|T| = 2$ this turns into the classical (undirected) min-cost max-flow problem.

In [Ka1] it was proved that $(*)$ has a *half-integral* optimal solution f, and that such an f can be found by a pseudo-polynomial algorithm. In [Ka2] a polynomial algorithm to find a half-integral optimal f was designed; however, it uses a variant of the ellipsoid method to solve the dual linear program.

In the present paper we develop two purely combinatorial polynomial-time algorithms for finding a half-integral optimal solution to $(*)$. One of them is based on capacity scaling, and the other one is based on cost scaling.

To design these algorithms, we introduce certain combinatorial structures, called transitive fork environments, and study general properties of such structures.

♯ Supported in part by ONR Young Investigator Award N00014-91-J-1855, NSF Presidential Young Investigator Grant CCR-8858097 with matching funds from AT&T, DEC, and 3M, and a grant from Powell Foundation.

♮ This research was done while the second author was visiting Stanford University Computer Science Department and supported by the above mentioned NSF and Powell Foundation Grants.

1. Introduction

Throughout, by a *graph* we mean a finite undirected graph without loops; VG is the *vertex-set* and EG is the *edge-set* of a graph G.

Let $N = (G, T, c, a)$ be a *network* consisting of a graph G, a subset $T \subseteq VG$ called the set of *terminals*, and nonnegative integer-valued functions $c : EG \to \mathbb{Z}_+$ (of edge *capacities*) and $a : EG \to \mathbb{Z}_+$ (of edge *costs*). We denote $|VG|$ by n.

Let $\mathcal{P} := \mathcal{P}(G, T)$ denote the set of simple paths from s to t, or $s-t$ paths, in G for all distinct $s, t \in T$. A (c-admissible) *multicommodity flow*, or a *multiflow*, in N is a function $f : \mathcal{P} \to \mathbb{Q}_+$ satisfying the capacity constraint

$$(1.1) \qquad \zeta^f(e) := \sum(f(P) \,:\, e \in P \in \mathcal{P}) \leq c(e) \quad \text{for all} \;\; e \in EG$$

(when writing $e \in P$, we consider a path as an edge-set). The *total value* v_f of f is $\sum(f(P) \,:\, P \in \mathcal{P})$, and the *total cost* a_f of f is $\sum(a(e)\zeta^f(e) \,:\, e \in EG)$. f is called *maximum* if v_f is as large as possible. The *minimum cost maximum multiflow* problem that we study in the paper is:

(1.2) *find a maximum multiflow f in N such that a_f is as small as possible.*

When $|T| = 2$, (1.2) turns into the classical (undirected) min-cost max-*flow* problem [FF], and it has an optimal solution f that is integer-valued. Simple examples show that such a property, in general, does not remain valid whenever $|T| \geq 3$. Nevertheless, the following is true.

Theorem 1 [Ka1]. *For $|T| \geq 3$, (1.2) has an optimal solution f that is half-integral, i.e., $2f$ is integer-valued.*

Instead of (1.2), it is convenient to deal with a slightly more general parametric problem; namely:

(1.3) *given $p \in \mathbb{Q}_+$, find a multiflow f in N which maximizes the objective function $pv_f - a_f$.*

Obviously, (1.3) is equivalent to (1.2) when p becomes large enough (in fact, one can show that taking p to be $2a(EG)c(EG)+1$ is sufficient). [For a mapping $g : S \to \mathbb{R}$ and a subset $S' \subseteq S$, $g(S')$ denotes $\sum(g(e) \,:\, e \in S')$.] So Theorem 1 is immediately implied by the following result.

Theorem 2 [Ka1]. *For any $p \geq 0$, (1.3) has a half-integral optimal solution.*

This result is a consequence of a pseudo-polynomial algorithm for (1.3) developed in [Ka1]; its running time is $O(\min\{(c(EG)+1)Q_1, (a(EG)+1)Q_2, 2^{Q_3}\})$, where Q_1, Q_2, Q_3 are polynomials in n.

A rather simple, though non-algorithmic, proof of Theorem 2 was given in [Ka2]. Also a strongly polynomial algorithm to find a half-integral optimal solution to (1.3) was developed there; however, that algorithm is not purely combinatorial as it uses a variant of the ellipsoid method to solve the linear program dual to (1.3).

In the present paper we design two purely combinatorial polynomial algorithms to find a half-integral solution to (1.2): an algorithm of complexity $O(P_1 \log(c(EG)+1))$ based on capacity scaling techniques (cf. [EK]); and an algorithm of complexity $O(P_2 \log(a(EG)+1))$ based on cost scaling techniques (cf. [Ro,BJ]; here P_1 and P_2 are polynomials in n.

This paper is organized as follows. In Section 2 we introduce some basic notions and discuss their properties. First we state the linear program dual to (1.3) and specify the optimality criterion for both problems. In particular, this criterion requires that any optimal multiflow go along shortest paths of a certain weighted subgraph Γ of G.

It should be noted that the approaches in [Ka1,Ka2] are based on the observation that in order to obtain the required multiflow f in Γ it suffices to find a certain one-commodity flow g in a digraph, called the double covering digraph over Γ; moreover, the existence of an integral g immediately implies that the "image " f of g in Γ is half-integral (whence Theorems 2 and 1 follow).

In contrast, approaches developed in the present paper are based on other ideas (some of them come from [Ka3] where the integral version of (1.2) is solved). In Sections 2-4 we show that, instead of multiflows in Γ, one can successfully operate directly with their total flow functions ζ^f (as in (1.1)) on $E\Gamma$. A function h of this kind, called *regular*, has a simple characterization (independent of a multiflow f behind h). It turns out that an optimal regular function can be obtained by use of transformations along "augmenting paths", which have a slightly more exotic form (somewhat like self-intersecting alternating paths arising in problems on fractional matchings) than usual augmenting paths in the theory of single commodity flows.

In order to get all necessary results concerning regular functions, augmenting paths, and transformations of dual solutions, we study, in Section 3, certain combinatorial structures, called *transitive fork environments.* As an illustration, we show that a simple transitive fork environment naturally arises in connection with problems on fractional b-matchings.

In Section 4 we show how to apply results on transitive fork environments to design a pseudo-polynomial algorithm for finding a half-integral optimal solution to (1.2); in particular, this provides an alternative proof of Theorem 2. Moreover, some elements of this algorithm give rise to constructing polynomial algorithms for (1.2); namely, the above-mentioned capacity scaling algorithm (Section 5) and cost-scaling algorithm (Section 6).

Although we allow G to have parallel edges, when it is not confusing, an edge with ends x and y may be denoted by xy. A part of a path $P=(x_0,e_1,x_1,\dots,e_k,x_k)$ from x_i to x_j $(i\le j)$ is denoted by $P\langle x_i,x_j\rangle$; and P^{-1} stands for the reverse path $(x_k,e_k,x_{k-1},\dots,e_1,x_0)$.

2. Duality, l-graphs, and regular functions

The linear program dual to (1.3) is

(2.1) minimize $c\gamma := \sum(c(e)\gamma(e)\ :\ e\in EG)$, provided that

(i) $\gamma\in\mathbb{Q}_+^{EG}$, and

(ii) $\gamma(P)\ge p-a(P)$ for any $P\in\mathcal{P}$

(considering P as an edge-set). For $l\in\mathbb{Q}_+^{EG}$, let $\mathrm{dist}_l(x,y)$ denote the *l-distance* between vertices $x,y\in VG$, i.e., the minimum *l-length* $l(P)$ of an $x-y$ path P in G. The system (2.1)(ii) can be rewritten in a more compact form, namely,

$$\mathrm{dist}_{a+\gamma}(s,t)\ge p \qquad \text{for any } s,t\in T,\ s\ne t. \tag{2.2}$$

The l.p. duality theorem applied to (1.3) and (2.1) implies that a (c-admissible) multiflow f and a vector $\gamma\in\mathbb{Q}_+^{EG}$ satisfying (2.2) are optimal solutions to these problems if and only if the following "complementary slackness" conditions hold:

(2.3) for $P\in\mathcal{P}$, if $f(P)>0$ then $a(P)+\gamma(P)=p$; in particular, P is an $(a+\gamma)$-*shortest* path in G;

(2.4) for $e\in EG$, if $\gamma(e)>0$ then e is *saturated* by f, i.e., $\zeta^f(e)=c(e)$.

Consider a *positive* function l on EG, i.e., $l(e)>0$ for all $e\in EG$. Put

$$p' := p^l := \min\{\mathrm{dist}_l(s,t)\ :\ s,t\in T, s\ne t\}. \tag{2.5}$$

A path P connecting different terminals and having l-length exactly p' is called a *T,l-line*, or, briefly, a *T-line*. Note that the positivity of l implies that any l-shortest path, in particular, a T-line, is simple. The subgraph of G whose edges belong to T-lines and vertices belong to T-lines or T is called the *l-graph* and denoted by $\Gamma = \Gamma^l$. The *potential* $\pi(v) := \pi_\lambda(v)$ of $v \in VG$ is the l-distance from v to T, i.e., $\min\{\mathrm{dist}_l(v,s) : s \in T\}$. Denote by $T(v)$ the set of terminals $s \in T$ closest to v, i.e., with $\mathrm{dist}_l(s,v) = \pi(v)$.

The vertices in Γ are naturally partitioned into the sets V_s ($s \in T$) and $V^\bullet$. Here $V_s := \{v \in V\Gamma : \mathrm{dist}_l(s,v) < p'/2\}$ and $V^\bullet := \{v \in V\Gamma : \pi(v) = p'/2\}$; a vertex in $V^\bullet$ is called *central*. Clearly $T(v) = \{s\}$ for $v \in V_s$, and $|T(v)| \geq 2$ for $v \in V^\bullet$. The following is easy (cf. [Ka2]):

(2.6) (i) an $x-y$ path P in G with $x, y \in V_s \cup V^\bullet$ for some $s \in T$ is a part of a T-line if and only if $l(P) = |\pi(x) - \pi(y)|$;

(ii) an $x-y$ path P in G with $x \in V_s$ and $y \in V_t$ for distinct $s,t \in T$ is a part of a T-line if and only if $\pi(x) + l(P) + \pi(y) = p'$.

[Note that, in view of the positivity of l, no edge in Γ can connect two central vertices, and $\pi(v) \neq \pi(z)$ for any edge $e = vz \in E\Gamma$ with $v, z \in V_s \cup V^\bullet$.]

For $v \in V\Gamma$ denote by $E(v)$ the set of edges in Γ incident to v. It will be useful for us to think that the terminals are numbered by the integers from 1 through $|T|$. To each $v \in V\Gamma$ and $e = vz \in E(v)$ we assign the *attachment* $s(x,e)$ by the following rule:

(2.7) (i) if $v \in V_s \cup V^\bullet$, $z \in V_s$ and $\pi(z) < \pi(v)$, put $s(v,e) := s$;

(ii) if $v \in V_s$ and either $z \notin V_s$, or $z \in V_s$ and $\pi(z) > \pi(v)$, put $s(v,e) := -s$.

One can see that if P is an arbitrary T-line from s to t which passes v, e, z (in this order) then $s(v,e) > 0$ implies $s(v,e) = t$, while $s(v,e) < 0$ implies $-s(v,e) = s$. Let $\langle T\rangle$ denote the set of integers $s \neq 0$ such that $-|T| \leq s \leq |T|$; then $\langle T\rangle$ is the set of all possible attachments for (v,e). For $v \in V\Gamma$ and $s \in \langle T\rangle$ define

$$E_s(v) := \{e \in E(v) : s(v,e) = s\}.$$

To illustrate this definition, consider two possible cases for $v \in V\Gamma$.

(C1) $v \in V_s$ for some $s \in T$. Then each edge $e = vz \in E\Gamma$ belongs to either $E_s(v)$ or $E_{-s}(v)$ (in the former case, $z \in V_s$ and $\pi(z) < \pi(v)$, while in the latter case, either $z \notin V_s$, or $z \in V_s$ and $\pi(z) > \pi(v)$).

(C2) $v \in V^\bullet$. Then for each $e = vz \in E\Gamma$, $z \in V_s$ for some $s \in T(v)$, whence e belongs to $E_s(v)$.

Using (2.6), it is easy to obtain the following characterization of the lines in terms of the attachments (cf. [Ka3]):

(2.8) A path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ is a part of a T-line if and only if for $i = 1 \dots, k-1$, the numbers $s(x_i, e_i)$ and $s(x_i, e_{i+1})$ are different.

Next, we say that a multiflow f in N *goes along T-lines* if for any $P \in \mathcal{P}$, $f(P) > 0$ implies that P is a T-line. (Note that if $\gamma \in \mathbb{Q}_+^{EG}$, $l = a + \gamma$ and $p = p^l$, then the property that a multiflow f goes along T, l-lines reflects the fact that f and γ satisfy the complementary slackness condition (2.3).) An important property, which immediately follows from (2.8), is that

(2.9) if f goes along T-lines then the inequality $\zeta^f(E_s(v)) \le \frac{1}{2}\zeta^f(E(v))$ holds for any $v \in V\Gamma - T$ and $s \in \langle T \rangle$.

It turns out that a property converse, in a sense, to (2.9) takes place. More precisely, let $h : E\Gamma \to \mathbb{Q}_+$ be a function.

Definition 2.1. h is called *regular* if h is *non-excessive* for each $v \in V\Gamma - T$; this means that

$$h(E_s(v)) \le \frac{1}{2} h(E(v)) \qquad \text{for any } \ s \in \langle T \rangle. \tag{2.10}$$

Definition 2.2. h is called *half-Eulerian* (*inner half-Eulerian*) if h is half-integral and $h(E(v))$ is an integer for any $v \in V\Gamma$ (respectively, $v \in V\Gamma - T$).

We say that $s \in \langle T \rangle$ is *tight* for v (and h) if the inequality in (2.10) holds with equality. The following statement, similar to one in [Ka3], enables us to handle regular functions rather than multiflows (here for a path P in G, χ^P denotes its incidence vector in $\mathbb{R}^{EG}$, i.e. $\chi^P(e)$ is 1 if $e \in P$ and 0 otherwise).

Statement 2.3. *If $h : E\Gamma \to \mathbb{Q}_+$ is regular then h is representable in the form $h = \lambda_1 \chi^{P_1} + \ldots + \lambda_m \chi^{P_m}$, where $\lambda_1, \dots, \lambda_m$ are positive rationals and $P_1, \dots, P_m$ are T-lines. Moreover, if, in addition, h is inner half-Eulerian then there exists a representation with all λ_i's half-integral.*

Proof. It suffices to prove the second part of the statement. We proceed by induction on $h(E\Gamma)$. The statement is trivial if $h = 0$; so we suppose that $e = xy$ is an edge with $h(e) > 0$. Consider a *maximal* path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ such that:

(2.11) (i) P contains e;

(ii) $h(e_i) > 0$ for $i = 1, \dots, k$;

(iii) for $i = 1, \dots, k-1$, $x_i \in V\Gamma - T$ and $s(x_i, e_i) \neq s(x_i, e_{i+1})$;

(iv) for $i = 1, \dots, k-1$, no $q \in \langle T \rangle - \{s(x_i, e_i), s(x_i, e_{i+1})\}$ is tight for x_i.

Such a P exists, as the path (x, e, y) satisfies (2.11). Note also that P is a part of a T-line (by (2.11)(iii) and (2.8)); in particular, P is simple.

We assert that P is a T-line, i.e., $x_0, x_k \in T$. Suppose, for a contradiction, that $x_k \notin T$ (case $x_0 \notin T$ is symmetric); let $v := x_k$, $u := e_k$, and $s := s(v, u)$. By (2.10) and the fact that $h(u) > 0$, one can choose $u' = vz \in E(v) - E_s(v)$ with $h(u') > 0$. Observe that no two different $s', s'' \in \langle T \rangle - \{s\}$ exist such that both s', s'' are tight for v (for otherwise $E_{s'} \cap E_{s''} = \emptyset$ and $u \notin E_{s'}, E_{s''}$ would imply $h(E_{s'}(v)) \geq h(E_{s''}(v)) + h(u)$ and $h(E_{s''}(v)) \geq h(E_{s'}) + h(u)$, which is impossible since $h(u) > 0$). Hence, u' can be chosen so that no $q \in \langle T \rangle$ is tight for v and different from s and $s(v, u')$. Then P extended by u, z satisfies (2.11), contrary to the maximality of P.

Thus, P is a T-line. Now define ε to be the minimum of ε_i $(i = 1, \dots, k-1)$, where

$$\text{(2.12)} \qquad \begin{aligned} \varepsilon_i := \min\{&h(e_i), h(e_{i+1}), \\ &\frac{1}{2}\min\{h(E(x_i)) - h(E_s(x_i)) \,:\, s \in \langle T \rangle, s \neq s(x_i, e_i), s(x_i, e_{i+1})\}\}, \end{aligned}$$

and reduce h to $h' := h - \varepsilon \chi^P$. From (2.12) it follows that h' is nonnegative and inner half-Eulerian (since h is inner half-Eulerian, whence $h(E(x_i)) - h(E_s(x_i))$ is an integer for $i = 1, \dots, k-1$ and $s \in \langle T \rangle$). Furthermore, $\varepsilon > 0$, by (2.11)(ii),(iv). Hence, $h'(E\Gamma) < h(E\Gamma)$, and now the result easily follows by induction. •

Remark 2.4. The above proof shows that the representation as required in Statement 2.3 can be found in strongly polynomial time. Indeed, when coming from h to h' as above, at least one of the following occurs, by (2.12): $h'(u) = 0$ for some $u \in E\Gamma$ with $h(u) > 0$, or there are $v \in V\Gamma - T$ and $s \in \langle T \rangle$ such that s is tight for v and h' but not for v and h. Thus, the number of iterations of the algorithm behind the above proof does not exceed $|E\Gamma| + 2|T|(|V\Gamma - T|)$ (taking into account that if for some $s \in \langle T \rangle$ and $v \in V\Gamma - T$, s is tight for v, h, then s is obviously tight for v, h').

One can see that if f is a multiflow arising from a regular h by use of a *decomposition* $\{(P_i, \lambda_i) \,:\, i = 1, \dots, m\}$ as in Statement 2.3 (i.e., $f(P_i) := \lambda_i$ for $i = 1, \dots, m$ and $f(P) := 0$ for the other P's in $\mathcal{P}(G, T)$), then

$$(2.13) \qquad v_f = \frac{1}{2}\sum_{s\in T}\sum(h(e)\,:\,e\in E(s)).$$

3. Forks and augmenting paths

Generalizing some ideas from [Ka3], in this section we introduce the notion of a transitive fork environment and establish some general results about it.

Suppose we are given a graph G', two subsets $T', T'' \subseteq VG'$ of *terminals* such that $T' \subseteq T''$, a function $h : EG' \to \mathbb{Q}_+$, and a set Π of tuples $\tau = (v, e, e', \sigma, \sigma')$, where $v \in VG' - T''$; $e, e' \in E_v$ (possibly $e = e'$); and $\sigma, \sigma' \in \{-1, 1\}$ (E_v is the set of edges in G' incident to v). A member τ of Π is called a *fork*. For $e \in EG'$ let $S(e) = S(e, h)$ denote the set $\{-1, 1\}$ if $h(e) > 0$, and $\{1\}$ if $h(e) = 0$. We say that Π is *transitive fork environment* for G', T'', h if is satisfies the following axioms:

(3.1) (i) $(v, e, e', \sigma, \sigma') \in \Pi$ implies $(v, e', e, \sigma', \sigma) \in \Pi$ (*symmetry*);

(ii) for $v \in VG' - T''$, $e, e' \in E_v$, and $\sigma \in S(e)$, at least one of $(v, e, e', \sigma, 1)$ and $(v, e, e', \sigma, -1)$ is in Π (*connectedness*);

(iii) $(v, e, e', \sigma, \sigma'), (v, e', e'', -\sigma', \sigma'') \in \Pi$ implies $(v, e, e'', \sigma, \sigma'') \in \Pi$ (*transitivity on forks*);

(iv) for $v \in VG' - T''$, $e_i \in E_v$, $\sigma_i \in \{-1, 1\}$, $i = 1, 2, 3$, if $\sigma_2 \in S(e_2)$ and $(v, e, e', \sigma, \sigma'), (v, e', e'', \sigma', \sigma'') \notin \Pi$ then $(v, e, e'', \sigma, \sigma'') \notin \Pi$ (*transitivity on non-forks*).

The *value* of h is defined to be

$$(3.2) \qquad v_h = \frac{1}{2}\sum(h(E_s)\,:\,s \in T'')$$

(cf. (2.13)). Let $c', c'' : EG' \to \mathbb{Z}_+$ be two functions such that $c' \le h \le c''$.

Consider a path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in G' and a sequence $\Sigma = (\sigma_1, \dots, \sigma_k)$, where $x_1, \dots, x_{k-1} \notin T''$ and $\sigma_i \in \{-1, 1\}$. The pair (P, Σ) is called a *signed* path, and σ_i the *sign* of e_i. We say that (P, Σ) is

(3.3) (i) a *feasible* path if for $i = 1, \dots, k-1$, $(x_i, e_i, e_{i+1}, \sigma_i, \sigma_{i+1}) \in \Pi$, and for $j = 1, \dots, k$, $\sigma_j > 0$ implies $h(e_j) < c''(e_j)$, and $\sigma_j < 0$ implies $h(e_j) > c'(e_j)$;

(ii) an *active* path if it is feasible, $x_0 \in T'$, and either $k = 0$, or $k \ge 1$ and $\sigma_1 = 1$;

(iii) an *augmenting* path if it is active, $k \geq 1$, $x_k \in T''$, and $\sigma_k = 1$.

Note that if (P, Σ) is feasible then, obviously, $\sigma_i \in S(e_i)$ for $i = 1, \ldots, k$. Define the incidence vector $\chi^{P,\Sigma} \in \mathbb{R}^{EG'}$ of (P, Σ) by

$$\chi^{P,\Sigma}(e) := \sum (\sigma_i \,:\, 1 \leq i \leq k, e = e_i) \qquad \text{for } \ e \in EG'. \tag{3.4}$$

The term "augmenting path" for (P, Σ) is motivated by the property that if we transform h to $h' := h + \varepsilon\chi^{P,\Sigma}$ with a sufficiently small $\varepsilon > 0$ then the new function h' is c', c''-admissible (i.e., $c' \leq h' \leq c''$) and has the value greater than that of h ($v_{h'} = v_h + 2\varepsilon$).

Example. This is from the matching theory. The fractional capacitated maximum b-matching problem is (see, e.g., [LP]): $(*)$: given functions $b : VG' \to \mathbb{Z}_+$ and $c', c'' : EG' \to \mathbb{Z}_+$ ($c' \leq c''$), find a function $h : EG' \to \mathbb{Q}_+$ such that $h(EG)$ is maximum, subject to (i) $c' \leq h \leq c''$, and (ii) $h(E_v) \leq b(v)$ for all $v \in VG'$. For h satisfying (i)-(ii), define $T' = T'' := \{v \in VG' \,:\, h(E_v) < b(v)\}$. Let Π consists of all $\tau = (v, e, e', \sigma, \sigma')$ such that $v \in VG' - T''$, $e, e' \in E_v$ and $\sigma = -\sigma'$. It is easy to see that Π satisfies (3.1), and the ε-transformation along an augmenting path as above increases the objective function. It is known that problem $(*)$ has a half-integral optimal solution h if it has a solution. Moreover, such an h can be found by use of $\frac{1}{2}$-transformations along special augmenting paths as in (3.5) (or (3.9)) below.

In Sections 4-6 we shall deal with other cases of transitive fork environments that will be used in algorithms to solve (1.2).

Now we study properties of active and augmenting paths. For $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ let $(P, \Sigma)^{-1}$ denote the reverse signed path ($P^{-1} = (x_k, e_k, x_{k-1}, \ldots, e_1, x_0)$, $\Sigma^{-1} = (\sigma_k, \ldots, \sigma_1)$); for $0 \leq i \leq j \leq k$ let $(P, \Sigma)\langle x_i, x_j\rangle$ denote the part $(P\langle x_i, x_j\rangle, \Sigma\langle \sigma_i, \sigma_j\rangle := (\sigma_i, \ldots, \sigma_j))$ of (P, Σ) from x_i to x_j.

Statement 3.1. *Suppose that there is an augmenting path for $G', T', T'', h, \Pi, c', c''$. Then there is an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ of at least one of the following forms:*

(3.5) (i) *P is a simple path or a simple circuit;*

(ii) *k is an even, $m = k/2$, $P\langle x_0, x_m\rangle$ is simple and $(P, \Sigma)\langle x_m, x_k\rangle$ is the reverse to $(P, \Sigma)\langle x_0, x_m\rangle$;*

(iii) *there is $0 < m < k/2$ such that $P\langle x_0, x_m\rangle$ is simple, $(P, \Sigma)\langle x_{k-m}, x_k\rangle$ is the reverse to $(P, \Sigma)\langle x_0, x_m\rangle$, and $P\langle x_m, x_{k-m}\rangle$ is a simple circuit disjoint from*

$P\langle x_0, x_m\rangle - \{x_m\}$.

(See Fig. 3.1.)

Proof. Consider an augmenting path $(P = (x_0, e_1, x_1, \dots, e_k, x_k), \Sigma = (\sigma_1, \dots, \sigma_k))$. We may assume that if $x_i = x_j$ for some $0 < i < j < k$ then $(x_i, e_i, e_{j+1}, \sigma_i, \sigma_{j+1}) \notin \Pi$; for otherwise we could replace (P, Σ) by the smaller augmenting path that is the concatenation of $(P, \Sigma)\langle x_0, x_i\rangle$ and $(P, \Sigma)\langle x_j, x_k\rangle$.

Suppose that P is not a simple path or a simple circuit. Let j be the maximum index such that $P' := P\langle x_0, x_j\rangle$ is a simple path. Then $v := x_{j+1}$ coincides with x_i for some $0 < i < j$, and $v \notin T'$. By the above argument, $\tau = (v, e_i, e_{j+2}, \sigma_i, \sigma_{j+2}) \notin \Pi$. Considering τ and $\tau' = (v, e_{j+1}, e_{j+2}, \sigma_{j+1}, \sigma_{j+2}) \in \Pi$, we deduce from (3.1)(iv) that $(v, e_i, e_{j+1}, \sigma_i, \sigma_{j+1}) \in \Pi$. Hence, the concatenation (P', Σ') of $(P, \Sigma)\langle x_0, x_{j+1}\rangle$ and the reverse to $(P, \Sigma)\langle x_0, x_i\rangle$ is an augmenting path. If $e_{i+1} \neq e_{j+1}$ then (P', Σ') is of the form as in (iii) of (3.5). Suppose that $e_{i+1} = e_{j+1} =: u$. Then $j = i + 1$, by the choice of j. If $\sigma_{i+1} = \sigma_{i+2}$ then (P', Σ') is as in (3.5)(ii). Finally, if $\sigma_{i+1} = -\sigma_{i+2}$ then applying (3.1)(iii) to $(v, e_i, e_{i+1}, \sigma_i, \sigma_{i+1}) \in \Pi$ and $(v, e_{i+3}, e_{i+2}, \sigma_{i+3}, \sigma_{i+2}) \in \Pi$ we conclude that $(v, e_i, e_{i+3}, \sigma_i, \sigma_{i+3}) \in \Pi$; a contradiction. •

(i) (ii) (iii)

Fig. 3.1

Note that Statement 3.1 can be slightly strengthened as:

(3.6) for (P, Σ) as in (3.5), $(x_q, e_q, e_q, \sigma_q, \sigma_q) \notin \Pi$ for $q = 1, \dots, k-1$ in case (i), for $q = 1, \dots, m-1$ in case (ii), and for $q = 1, \dots, m$ in case (iii).

(For otherwise, in (ii)-(iii), we could delete a part of (P, Σ) obtaining a smaller augmenting path of the form as in (ii); and in (i), if $(x_q, e_q, e_q, \sigma_q, \sigma_q) \in \Pi$ for some $0 < q < k$ then the concatenation of $(P, \Sigma)\langle x_0, x_q\rangle$ and its reverse is an augmenting path too.) Note also that the above arguments provide a polynomial subroutine that, given an augmenting path, finds an augmenting path as in (3.6).

Now we are interested in the special case when h is *inner half-Eulerian*, i.e., h is half-integral and $h(E_v)$ is an integer for each $v \in VG' - T''$ (cf. Definition 2.2). Let $H = H^h$ be the subgraph of G' induced by the edges $e \in EG'$ with $h(e)$ half-integral but not integral. Then for each $v \in VH - T''$ the number of edges in H incident to v is even. Hence, H is the union of pairwise edge-disjoint simple circuits C and simple paths (or circuits) P such that every C does not meet T'', while every P meets T'' in its end vertices (which can coincide) and only them. We say that C (P) is an *inner* $\frac{1}{2}$*-circuit* (respectively, a $\frac{1}{2}$*-path*) for G', T'', h. We need three procedures which eliminate, when possible, such circuits and paths. Recall that c', c'' are integer-valued, therefore, for $u \in EH$ we have $c'(u) < h(u) < c''(u)$, in particular, $S(u) = \{-1, 1\}$.

For a signed path (P, Σ), we say that the function $h' := h + \frac{1}{2}\chi^{P,\Sigma}$ is obtained from h by the $\frac{1}{2}$-transformation along (P, Σ), and denote this by $h \xrightarrow{P,\Sigma} h'$, where $\chi^{P,\Sigma}$ is defined in (3.4).

(A) *Elimination of a* $\frac{1}{2}$*-path.* Let $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ be a $\frac{1}{2}$-path. Going along P we form a feasible path $(P, \Sigma = (\sigma_1, \ldots, \sigma_k))$ as follows. Start with $\sigma_1 = 1$. Suppose that $\sigma_1, \ldots, \sigma_i$ have been already determined for some $i < k$. By (3.1)(ii), there is $\sigma \in \{-1, 1\}$ such that $(x_i, e_i, e_{i+1}, \sigma_i, \sigma) \in \Pi$ (since $\sigma_i \in S(e_i)$). Then put $\sigma_{i+1} := \sigma$; and so on. For the resulting (P, Σ) we make the transformation $h \xrightarrow{P,\Sigma} h'$. Note that $\sigma(e_1) = 1$ implies that $v_{h'} = v_h + \frac{1}{2}$ if $\sigma(e_k) = 1$, and $v_{h'} = v_h$ if $\sigma(e_k) = -1$. Furthermore, h' is inner half-Eulerian, and $h'(e)$ is integral for all edges e of P.

(B) *Conditional elimination of a* $\frac{1}{2}$*-path.* This operation will be used in Sections 5-6. It applies to a $\frac{1}{2}$-path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in the special case when $x_0 \in T'$, $x_k \in T'' - T'$, and it is not desirable to decrease $\sum(h(E_s) : s \in T'' - T')$. As above, we form a feasible path $(P, \Sigma = (\sigma_1, \ldots, \sigma_k))$ with $\sigma_1 = 1$. If we finish with $\sigma_k = 1$, we make the transformation $h \xrightarrow{P,\Sigma} h'$, which increases both values $h(E_{x_0})$ and $h(E_{x_k})$ by 1/2. Otherwise we examine the pairs e_i, e_{i+1} for $i = 1, \ldots, k-1$. Suppose that $(x_i, e_i, e_{i+1}, \sigma_i, -\sigma_{i+1})$ is a fork for some i. By (3.1)(ii) for $e := e_{i+2}$, $e' := e_{i+1}$ and $\sigma := -\sigma_{i+2} \in S(e)$, at least one of $(x_{i+1}, e', e, \sigma_{i+1}, \sigma)$ and $(x_{i+1}, e', e, -\sigma_{i+1}, \sigma)$ is a fork. Therefore, there are $\sigma'_1, \ldots, \sigma'_{i+2}$ with $\sigma'_1 = 1$ and $\sigma'_{i+2} = -\sigma_{i+2}$ such that $(P\langle x_0, x_{i+2}\rangle, (\sigma'_1, \ldots, \sigma'_{i+2}))$ is feasible. Repeating this argument we conclude that there is $\Sigma'' = (\sigma''_1, \ldots, \sigma''_k)$ with $\sigma''_1 = 1$, and $\sigma''_k = -\sigma_k = 1$ such that (P, Σ'') is a feasible path. Then we make the transformation $h \xrightarrow{P,\Sigma''} h'$.

(C) *Elimination of an inner* $\frac{1}{2}$*-circuit.* Let $C = (x_0, e_1, x_1, \ldots, e_k, x_k = x_0)$ be an inner $\frac{1}{2}$-circuit. As above, we form a feasible path $(C, \Sigma = (\sigma_1, \ldots, \sigma_k))$. Then we examine σ_1, σ_k appeared. If $\tau = (x_0, e_1, e_k, \sigma_1, \sigma_k) \in \Pi$ then we make the transformation $h \xrightarrow{C,\Sigma} h'$. If not, we examine the pairs e_i, e_{i+1} ($i = 1, \ldots, k-1$); as above, if some $(x_i, e_i, e_{i+1}, \sigma_i, -\sigma_{i+1})$ is a fork, we conclude that there is $\Sigma'' = (\sigma''_1, \ldots, \sigma''_k)$ with

$\sigma_1'' = \sigma_1$ and $\sigma_k'' = -\sigma_k$ such that (C, Σ'') is a feasible path. Now (3.1)(ii) and $\tau \notin \Pi$ imply that $(x_0, e_1, e_k, \sigma_1, -\sigma_k) \in \Pi$. Then we make the transformation $h \xrightarrow{C,\Sigma''} h'$.

The situation when we cannot eliminate half-integrality in cases (B) or (C) is:

(3.7) (i) if $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is a $\frac{1}{2}$-path with $x_0 \in T'$ and $x_k \in T'' - T'$ then there is a unique sequence $\Sigma = (\sigma_1, \ldots, \sigma_k)$ such that $\sigma_1 = 1$ and $(x_i, e_i, e_{i+1}, \sigma_i, \sigma_{i+1}) \in \Pi$ for $i = 1, \ldots, k-1$; and for this Σ, $\sigma_k = -1$ occurs; moreover, $(-\sigma_1, \ldots, -\sigma_k)$ is the unique sequence such that $\sigma_1 = -1$, $\sigma_k = 1$ and $(x_i, e_i, e_{i+1}, -\sigma_i, -\sigma_{i+1}) \in \Pi$ for $i = 1, \ldots, k-1$;

(ii) if $C = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is an inner $\frac{1}{2}$-circuit then, up to multiplying all signes by -1, there is a unique $\Sigma = (\sigma_1, \ldots, \sigma_k)$ such that $(x_i, e_i, e_{i+1}, \sigma_i, \sigma_{i+1} \in \Pi$ for $i = 1, \ldots, k-1$; and for this Σ, $(x_0, e_1, e_k, \sigma_1, \sigma_k) \notin \Pi$.

Such P and C are called *non-eliminatable*. Note that the initial vertex x_0 in C can be chosen arbitrarily.

(D) *Elimination of two touching inner $\frac{1}{2}$-circuits.* Let $C = (x_0, e_1, x_1, \ldots, e_k, x_k)$ and $C' = (y_0, u_1, y_1, \ldots, u_m, y_m)$ be two edge-disjoint non-eliminatable inner $\frac{1}{2}$-circuits with $x_0 = y_0 =: v$. Let $(C, \Sigma = (\sigma_1, \ldots, \sigma_k))$ and $(C', \Sigma' = (\sigma_1', \ldots, \sigma_m'))$ be feasible paths. Since Σ can be considered up to a multiple of -1, we may assume that $\tau = (v, e_k, u_1, \sigma_k, \sigma_1') \in \Pi$. We observe that $\tau' = (v, u_m, e_1, \sigma_m', \sigma_1) \in \Pi$, for otherwise $\tau'' = (v, e_1, u_1, \sigma_1, \sigma_1') \notin \Pi$ (by (3.1(iv) for τ' and $(v, u_1, u_m, \sigma_1', \sigma_m')$). Then $\tau'' \notin \Pi$ and $(x, e_1, e_k, \sigma_1, \sigma_k) \notin \Pi$ imply $\tau \notin \Pi$; a contradiction.

We eliminate half-integrality on both C, C' by making the transformation $h \xrightarrow{C'',\Sigma''} h'$, where C'' (Σ'') is the concatenation of C and C' (respectively, Σ and Σ'). Similarly, one can eliminate half-integrality on a pair of non-eliminatable paths P, P' touching in an inner point, or a pair of touching a path P and inner $\frac{1}{2}$-circuit C.

The above elimination procedures motivate introducing the following classes of h's. Namely, for a half-integral h we say that:

(3.8) (i) h is *semi-perfect* if H^h consists of non-eliminatable inner circuits and paths from T' to $T'' - T'$ that are vertex-disjoint except, possibly, these paths have some terminals in common;

(ii) h is *perfect* if it is semi-perfect and H^h contains only (inner) circuits.

(We shall deal with perfect h's in Section 4, and with semi-perfect h's in Sections 5 and 6.) For a perfect h, (3.5)-(3.6) are strengthened as follows:

(3.9) there exists an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ such that: either

(i) all $x_1, \ldots, x_{k-1}$ are different; $(x_q, e_q, e_q, \sigma_q, \sigma_q) \notin \Pi$ for $q = 1, \ldots, k-1$; and P is disjoint from H^h; or

(ii) k is an even $2m$; $P\langle x_0, x_m\rangle$ is simple; $(P, \Sigma)\langle x_0, x_m\rangle$ is reverse to $(P, \Sigma)\langle x_m, x_k\rangle$; $(x_q, e_q, e_q, \sigma_q, \sigma_q) \notin \Pi$ for $q = 1, \ldots, m-1$; and P is disjoint from H^h; or

(iii) there is $0 < m < k/2$ such that $P\langle x_0, x_m\rangle$ is simple; $(P, \Sigma)\langle x_0, x_m\rangle$ is reverse to $(P, \Sigma)\langle x_{k-m}, x_k\rangle$; $P\langle x_m, x_{k-m}\rangle$ is a simple circuit disjoint from $P\langle x_0, x_m\rangle - \{x_m\}$; $(x_q, e_q, e_q, \sigma_q, \sigma_q) \notin \Pi$ for $q = 1, \ldots, m$; and if P has a common vertex with some circuit C in H^h then C coincides with $P\langle x_m, x_{k-m}\rangle$.

Indeed, consider an augmenting path (P, Σ) as in Statement 3.1 and (3.6), and suppose that a vertex x_q of P belongs to a non-eliminatable circuit $(C = (z_0, w_1, z_1, \ldots, w_r, z_r), \Sigma' = (\sigma'_1, \ldots, \sigma'_r))$ (considered as a feasible path). One may assume that q is minimal under this property, $x_q = z_0 =: v$, and $\tau = (v, e_q, w_1, \sigma_q, \sigma'_1) \in \Pi$. Since $\tau \in \Pi$ and $(v, w_1, w_r, \sigma'_1, \sigma'_r) \notin \Pi$, we have $(v, e_q, w_r, \sigma_q, \sigma'_r) \in \Pi$ (by (3.1)(iv)). Then the concatenation of $(P, \Sigma)\langle x_0, x_q\rangle$, (C, Σ') and the reverse to $(P, \Sigma)\langle x_0, x_q\rangle$ is augmenting, whence (3.9) easily follows.

As to the semi-perfect case, a slightly more difficult statement describing a special sort of augmenting paths takes place; it will be used in Sections 5-6.

Fig. 3.2

Statement 3.2. *Let h be semi-perfect, and let there exist an augmenting path from T' to $T'' - T'$. Then there exists an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ from T' to $T'' - T'$ satisfying the following conditions:*

(3.10) (i) *if $x_i = x_j$ for some $0 < i < j < k$ then $(x_i, e_i, e_{j+1}, \sigma_i, \sigma_{j+1}) \notin \Pi$;*

(ii) *there are no $0 < i < j < r < k$ such that $x_i = x_j = x_r$;*

(iii) *if $e_i = e_j$ for some $0 < i < j < k$ then $x_{i-1} = x_j$ and $\sigma_i = \sigma_j$;*

(iv) *if* $e_i = e_j$ *for some* $i \neq j$ *then* $e_i \notin EH^h$ *(i.e.* $h(e_i)$ *is an integer).*

(See Fig. 3.2)

Proof. Let $\mathcal{C}$ and $\mathcal{L}$ denote the sets of circuits and T' to $(T''-T')$ paths in H^h, respectively. For $L = (y_0, u_1, y_1, \dots, u_m, y_m) \in \mathcal{L}$ let Σ_L denote the sequence $(\sigma_1, \dots, \sigma_k)$ such that (L, Σ_L) is the feasible path with $\sigma_1 = 1$ (and $\sigma_k = -1$), and $-\Sigma_L$ denote the sequence $(-\sigma_1, \dots, -\sigma_k)$ (cf. (3.7)(i)). Similarly, for $C \in \mathcal{C}$ denote by Σ_C a sign sequence such that (C, Σ_C) is a feasible path (cf. (3.7)(ii)).

Consider an augmenting path $(P = (x_0, e_1, x_1, \dots, e_k, x_k), \Sigma = (\sigma_1, \dots, \sigma_k))$ from $x_0 \in T'$ to $x_k \in T'' - T'$. Suppose that P meets some $C = (y_0, u_1, y_1, \dots, u_m, y_m) \in \mathcal{C}$. Let i and j be the minimum and maximum indices, respectively, such that x_i and x_j are in C; we may assume that $x_i = y_0 =: v$ and $x_j = y_r =: z$ for some $0 \leq r < m$. Let $e := e_i$, $e' := e_{j+1}$, $\sigma := \sigma_i$, $\sigma' := \sigma_{j+1}$; and let $\Sigma_C := (\sigma'_1, \dots, \sigma'_m)$. Two cases are possible.

(i) $r > 0$. We may assume that $\tau = (v, e, u_1, \sigma, \sigma'_1) \in \Pi$. If $\tau' = (z, e', u_r, \sigma', \sigma'_r) \in \Pi$, we replace in (P, Σ) the part $(P, \Sigma)\langle x_i, x_j\rangle$ by $(C, \Sigma_C)\langle y_0, y_r\rangle$, obtaining again an augmenting path from T' to $T'' - T'$. Now suppose that $\tau' \notin \Pi$. Since $(z, u_r, u_{r+1}, \sigma'_r, \sigma'_{r+1}) \in \Pi$ and $\tau' \notin \Pi$, $\tau'' = (z, e', u_{r+1}, \sigma', \sigma'_{r+1}) \in \Pi$ (by (3.1)(iv)). Now $\tau \in \Pi$ and $(v, u_1, u_m, \sigma_1, \sigma_m) \notin \Pi$ imply $(v, e, u_m, \sigma, \sigma_m) \in \Pi$ (by (3.1)(iv)). Hence, replacing the part of (P, Σ) from x_i to x_j by the reverse to $(C, \Sigma_C)\langle y_r, y_m\rangle$ we obtain an augmenting path.

(ii) $r = 0$. If $\tau = (v, e, e', \sigma, \sigma') \in \Pi$, we remove from (P, Σ) the part from x_i to x_j, obtaining an augmenting path. Let $\tau \notin \Pi$. We may assume that $\tau' = (v, e, u_1, \sigma, \sigma'_1) \in \Pi$. Then $(v, u_1, u_m, \sigma'_1, \sigma'_m) \notin \Pi$ implies that $\tau'' = (v, e, u_m, \sigma, \sigma'_m) \in \Pi$ (by (3.1)(iv)). Now $\tau \notin \Pi$ and $\tau'' \in \Pi$ imply that $(v, e', u_m, \sigma', \sigma'_m) \in \Pi$. Hence, replacing in (P, Σ) the part from x_i to x_j by (C, Σ_C) results in an augmenting path.

In view of these arguments, we may assume that

(3.11) if P intersects some $C = (y_0, u_1, y_1, \dots, u_m, y_m) \in \mathcal{C}$ then this intersection forms a part $P\langle x_i, x_j\rangle$ of P and a part $C\langle y_0, y_r\rangle$ of C; moreover, $\{x_{i+1}, \dots, x_{j-1}\}$ disjoint from $\{x_0, \dots, x_i, x_j, \dots, x_k\}$.

Let i_0 (i_1) be the maximum (minimum) index such that $P_0 := P\langle x_0, x_{i_0}\rangle$ (resp. $P_1 := P\langle x_{i_1}, x_k\rangle$) is a part of some path in $\mathcal{L}$, or P_0 (P_1) consists of a single vertex. Note that P_0 and P_1 are simple, and $i_0 < i_1$ (otherwise $P \in \mathcal{L}$, and then $\sigma_1 = \sigma_k = 1$ contradicts (3.7)(i)). Assuming that (P, Σ) is chosen so that $\omega(P) := i_1 - i_0$ is as small as possible under (3.11), we now show that (3.10) is satisfied. Let τ_i denote the fork $(x_i, e_i, e_{i+1}, \sigma_i, \sigma_{i+1})$.

First of all, if $x_i = x_j$ and $(x_i, e_i, e_{j+1}, \sigma_i, \sigma_{j+1}) \in \Pi$ for some $i < j$ then removing the part $(P, \Sigma)\langle x_i, x_j\rangle$ results in an augmenting path satisfying (3.11) and having smaller ω (as P_0 and P_1 are simple). Thus, (i) in (3.10) is true. Next, the existence of $i < j < r$ with $x_i = x_j = x_r =: v$ is impossible. For otherwise $(v, e_i, e_{j+1}, \sigma_i, \sigma_{j+1}) \notin \Pi$ (by (i)) and $\tau_j \in \Pi$ would imply $\tau = (v, e_i, e_j, \sigma_i, \sigma_j) \in \Pi$ (by (3.1)(iv)), and now $\tau \in \Pi$ and $(v, e_i, e_{r+1}, \sigma_i, \sigma_{r+1}) \notin \Pi$ (by (i)) would imply $(v, e_j, e_{r+1}, \sigma_j, \sigma_{r+1}) \in \Pi$, contrary to (i) for j and r. Thus, (ii) is true.

Suppose that $e_i = e_j =: e$ for some $i < j$. If $x_i = x_j$ and $\sigma_i = \sigma_j$, remove the part $(P, \Sigma)\langle x_i, x_j\rangle$. If $x_i = x_j$ and $\sigma_i = -\sigma_j$ then $(x_{i-1}, e_{i-1}, e_{j-1}, \sigma_{i-1}, \sigma_{j-1}) \in \Pi$, by (3.1)(iii) (since $\tau_{r-1} \in \Pi$ for $r = i, j$, and $\sigma_i = -\sigma_j$). Similarly, $(x_i, e_{i+1}, e_{j+1}, \sigma_{i+1}, \sigma_{j+1}) \in \Pi$. Replace (P, Σ) by the concatenation of $(P, \Sigma)\langle x_0, x_{i-1}\rangle$, the reverse to $(P, \Sigma)\langle x_i, x_{j-1}\rangle$, and $(P, \Sigma)\langle x_j, x_k\rangle$. Finally, if $x_{i-1} = x_j =: v$ and $\sigma_i = -\sigma_j$ then $(v, e_{i-1}, e_{j+1}, \sigma_{i-1}, \sigma_{j+1}) \in \Pi$ (by (3.1)(iii)), and we remove the part $(P, \Sigma)\langle x_{i-1}, x_j\rangle$. It is easy to see that in each case we get an augmenting path from T' to $T'' - T'$ that satisfies (3.11) and has a smaller ω. Thus, (iii) is true.

Now suppose that $e_i = e_j =: e$ for some $i < j$, and e belongs to a path $L = (y_0, u_1, \ldots, y_m) \in \mathcal{L}$. Let $e = u_r$. By (iii), $x_{i-1} = x_j =: v$, $x_i = x_{j-1} =: z$ and $\sigma_i = \sigma_j =: \sigma$. Let $\Sigma_L = (\sigma'_1, \ldots, \sigma'_m)$; then $y_0 \in T'$, $u_m \in T'' - T'$ and $\sigma'_1 = 1$. Four cases are possible.

(a) $y_{r-1} = v$ and $\sigma'_r = -\sigma$. Then $(v, e_{j+1}, u_{r-1}, \sigma_{j+1}, \sigma'_{r-1}) \in \Pi$ (by (3.1)(iii) for $\tau_j \in \Pi$ and $(v, u_{r-1}, u_r, \sigma'_{r-1}, \sigma'_r) \in \Pi$). Replace (P, Σ) by the concatenation of $(L, \Sigma_L)\langle y_0, y_{r-1}\rangle$ and $(P, \Sigma)\langle x_j, x_k\rangle$.

(b) $y_{r-1} = v$ and $\sigma'_r = \sigma$. Take the feasible path $(L, -\Sigma_L)$; then $-\sigma_m = 1$. We have $(z, e_{j-1}, u_{r+1}, \sigma_{j-1}, -\sigma'_{r+1}) \in \Pi$ (by (3.1)(iii) for $\tau_{j-1} \in \Pi$ and $(z, u_{r+1}, u_r, -\sigma'_{r+1}, -\sigma'_r) \in \Pi$). Replace (P, Σ) by the concatenation of $(P, \Sigma)\langle x_0, x_{j-1}\rangle$ and $(L, -\Sigma_L)\langle y_r, y_m\rangle$.

(c) $y_r = v$ and $\sigma'_r = -\sigma$. This case is symmetric, in a sense, to (b). Replace (P, Σ) by the concatenation of $(L, \Sigma_L)\langle y_0, y_{r-1}\rangle$ and $(P, \Sigma)\langle x_i, x_k\rangle$.

(d) $y_r = v$ and $\sigma'_r = \sigma$. This case is symmetric, in a sense, to (a). Replace (P, Σ) by the concatenation of $(P, \Sigma)\langle x_0, x_{i-1}\rangle$ and $(L, -\Sigma_L)\langle y_r, y_m\rangle$.

One can check that in each of (a)–(d) we obtain an augmenting path from T' to $T'' - T'$ that satisfies (3.11) and has a smaller ω. This proves (iv). •

The proof of Statement 3.2 provides a polynomial procedure that, given an augmenting path from T' to $T''-T'$, finds an augmenting path from T' to $T''-T'$ satisfying (3.10).

Corollary 3.3. *Let h be semi-perfect, and let there exist an augmenting path. Then at least one of the following is true:*

(i) *there exists an augmenting path (P, Σ) satisfying (3.9) except that some end (or both ends) of P may belong to H^h;*

(ii) *there exists an augmenting path from T' to $T'' - T'$ as in (3.10).*

Proof. It suffices to consider the case when an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma)$ with $x_0, x_k \in T'$ satisfies (3.9) but P has a vertex x_i $(0 < i < k)$ in common with some T' to $(T'' - T')$ path $L = (y_0, u_1, y_1, \ldots, u_m, y_m)$ in H^h. Let $x_i = y_j =: v$. From (3.1)(iv) it follows that at least one of $\tau = (v, e_i, u_{j+1}, \sigma_i, -\sigma'_{j+1})$ and $\tau' = (x, e_{i+1}, u_{j+1}, \sigma_{i+1}, -\sigma'_{j+1})$ is a fork (where $\Sigma_L = (\sigma'_1, \ldots, \sigma'_m)$). Assuming for definiteness that $\tau \in \Pi$, form the augmenting path from T' to $T'' - T'$ to be the concatenation of $(P, \Sigma)\langle x_0, x_i\rangle$ and $(L, -\Sigma_L)\langle y_j, y_m\rangle$. Now apply Statement 3.2. •

In further sections for an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k)$ as in (3.9)-(3.10) we shall use the transformation $h \xrightarrow{P,\Sigma} h'$ defined as follows:

(3.12) (i) if all $x_1, \ldots, x_{k-1}$ are different, h is integral on P, and $(x_i, e_i, e_i, \sigma_i, \sigma_i) \notin \Pi$ for $i = 1, \ldots, k-1$ then set $h'(e_i) := h(e_i) + \sigma_i$ for $i = 1, \ldots, k$ and $h'(e) := h(e)$ for the other $e \in EG'$;

(ii) otherwise for $e \in EG'$ set $h'(e) := h(e) + \Sigma(\sigma_i/2 : i \in \{1, \ldots, k\},\ e = e_i)$.

Obviously, h' is perfect in case (3.12)(i) and h' is inner half-Eulerian (not necessarily semi-perfect) in case (3.12)(ii).

Return to the case of general h. We now associate with $G', T', T'', h, \Pi, c', c''$ the digraph $D = (VD, AD)$ along with numbers $\sigma^q \in \{-1, 1\}$ on the arcs $q \in AD$. Here VD is the set of vertices of G' that occur in active paths; in particular, $T' \subseteq VD$. Each edge $e = xy \in EG'$ for which there is an active path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ with $e = e_k$ and $y = x_k$ produce an arc $q = (x, y) \in AD$ with $\sigma^q = \sigma_k$; the edge e underlying q is denoted by e^q. A priori, $e = xy$ can derive at most two arcs q, q' from x to y (with $\sigma^q = 1$ and $\sigma^{q'} = -1$) and at most two arcs $\overline{q}, \overline{q}'$ from y to x (with $\sigma^{\overline{q}} = 1$ and $\sigma^{\overline{q}'} = -1$). Let A_v^+ (A_v^-) denote the set of arcs *entering* (resp. *leaving*) $v \in VD$.

Lemma 3.4. *Exactly one of two alternatives (a) and (b) is true:*

(a) *there exists an augmenting path;*

(b) *(D, σ) satisfies the following conditions:*

(3.13)(i) *D has no arc $q=(x,y)$ with $y \in T''$ and $\sigma^q = 1$;*

(ii) *each $e = xy \in EG'$ produces at most two arcs in D, and if q, q' are different arcs with $e^q = e^{q'} = e$ then they are oppositely oriented ($q = (x,y)$ and $q' = (y,x)$ say), and $\sigma^q = -\sigma^{q'}$;*

(iii) *let $v \in VD - T''$ and $u \in E_v$; then for any $q \in A_v^+$ there is a unique number $\sigma(v,u,q)$ such that $(v, e^q, u, \sigma^q, \sigma) \in \Pi$; moreover, these numbers are the same for all $q \in A_v^+$.*

Proof. First of all we observe that (a) and (i) in (3.13) cannot be simultaneously true. Suppose that no augmenting path exists. We have to show validity of (ii)-(iii) in (3.13).

To see (ii), suppose that q, q' are two arcs in D from x to y such that $e^q = e^{q'} =: e$ and $\sigma^q = -\sigma^{q'} =: \sigma$. Then there are active paths $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ and $(P' = (y_0, u_1, y_1, \ldots, u_m, y_m), \Sigma' = (\sigma'_1, \ldots, \sigma'_m))$ such that $e_k = u_m = e$, $x_k = y_m = y$, $\sigma_k = \sigma$ and $\sigma'_m = -\sigma$. Note that $x \notin T'$ (otherwise $\sigma_k = \sigma'_m = 1$, by (3.3)(ii)). Applying (3.1)(iii), we have $(x, e_{k-1}, u_{m-1}, \sigma_{k-1}, \sigma'_{m-1}) \in \Pi$. Hence, the concatenation of $(P,\Sigma)\langle x_0, x_{k-1}\rangle$ and the reverse to $(P',\Sigma')\langle y_0, y_{m-1}\rangle$ is an augmenting path (taking into account that $T' \subseteq T''$); a contradiction. Now suppose that $q \in AD$ goes from x to y; $q' \in AD$ goes from y to x; $e^q = e^{q'} =: e$; and $\sigma^q = \sigma^{q'} =: \sigma$. Choose active paths $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ and $(P' = (y_0, u_1, y_1, \ldots, u_m, y_m), \Sigma' = (\sigma'_1, \ldots, \sigma'_m))$ such that $e_k = u_m = e$, $x_k = y_{m-1} = y$ and $\sigma_k = \sigma'_m = \sigma$. Then the concatenation of (P,Σ) and the reverse to $(P',\Sigma')\langle y_0, y_{m-1}\rangle$ is augmenting; a contradiction.

To see (iii), consider $q \in A_v^+$ and an active path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ with $x_k = v$ and $e_k = e^q$. Then $\sigma_k = \sigma^q$, by (ii). Since $\sigma_k \in S(v)$, there is at least one fork $\tau = (v, e^q, u, \sigma^q, \sigma)$, by (3.1)(ii). If $\tau' = (v, e^q, u, \sigma^q, -\sigma)$ were a fork either, we would observe from (3.1)(iii) for τ, τ' that $(v, e^q, e^q, \sigma^q, \sigma^q) \in \Pi$. Then the concatenation of (P,Σ) and its reverse is an augmenting path; a contradiction. Next, suppose that $\sigma := \sigma(v,u,q) \neq \sigma(v,u,q')$ for some $q, q' \in A_v^+$. Choose active paths $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ and $(P' = (y_0, u_1, y_1, \ldots, u_m, y_m), \Sigma' = (\sigma'_1, \ldots, \sigma'_m))$ such that $e^q = e_k$, $e^{q'} = u_m$ and $v = x_k = y_m$. Applying (3.1)(iii), we obtain that $(v, e_k, u_m, \sigma_k, \sigma'_m) \in \Pi$, whence the concatenation of (P,Σ) and the reverse to (P',Σ') is an augmenting path; a contradiction. •

Now we study the case when no augmenting path exists. We say that the vertices and edges of G' that occur in active paths are *labeled*. Let D and σ be the digraph and mapping as above. For $v \in VD - T''$ and $u \in E_v$ the number $\sigma(v,u,q)$ will be denoted by $\sigma(v,u)$ (this number does not depend on $q \in A_v^+$, by (3.13)(iii)). We now introduce the following numbers, which will play an important role in transformation of dual solutions in our algorithms to solve (1.2):

$$(3.14)\qquad \text{for } v\in VG' \text{ and } e\in E_v \text{ put}\quad \begin{aligned} \rho(v,e) &:= 1 && \text{if } v\in T';\\ &:= 0 && \text{if } v\notin VD \text{ or } v\in T''-T';\\ &:= \sigma(v,e) && \text{if } v\in VD-T''; \end{aligned}$$

$$(3.15)\qquad \text{for } e=xy\in EG' \text{ put } \ \rho(e):=\rho(x,e)+\rho(y,e).$$

We observe that

(3.16) for any labeled edge $e=xy$:

(i) if $e=e^q$ for $q=(x,y)\in AD$ then $x\notin T'$ implies $\sigma(x,e)=\sigma^q$ and $y\notin T''$ implies $\sigma(y,e)=-\sigma^q$;

(ii) $\rho(e)=0$ unless some of x,y is in $T''-T'$.

Indeed, let $(P=(x_0,e_1,x_1,\dots,e_k,x_k),\Sigma=(\sigma_1,\dots,\sigma_k))$ be an active path with $x_k=y$ and $e_k=e$. If $x\notin T'$ (and therefore $x\in VG'-T''$) then $\sigma(x,e)=\sigma^q$ follows from $\sigma(x,e)=\sigma(x,e,q')$ for $q'=(x_{k-2},x)\in A^+_x$ and $(x,e_{k-1},e_k,\sigma_{k-1},\sigma_k=\sigma^q)\in\Pi$. Let $y\notin T''$. If $\sigma(y,e)=\sigma^q$ $(=\sigma_k)$ then the concatenation of (P,Σ) and its reverse would be an augmenting path; a contradiction. Thus, (i) is true. To see (ii), consider three cases. If $y\notin T''$ and $x=x_{k-1}\notin T'$ then, by (i) and (3.14), $\rho(y,e)=-\sigma^q$ and $\rho(x,e)=\sigma^q$, whence $\rho(e)=0$. If $y\notin T''$ and $x\in T'$ then, by the definition of an active path, $k=1$ and $\sigma^q=\sigma_k=1$, whence $\rho(y,e)=-1$ (by (i)). Now $\rho(x,e)=1$ (by (3.14)) implies $\rho(e)=0$. Finally, if $y\in T''$ then $\sigma^q=\sigma_k=-1$. Hence, $x\notin T'$, and we obtain $\rho(x,e)=-1$ (by (i)), and $\rho(y,e)=1$ if $y\in T'$.

Definition 3.5. A path $P=(x_0,e_1,x_1,\dots,e_k,x_k)$ with $k\ge 1$ is called a $(+)$-T'-*line* (resp. a $(+)$-T'-T''-*line*) if: (i) $x_0,x_k\in T'$ (resp. $x_0\in T'$, $x_k\in T''-T'$) and $x_1,\dots,x_{k-1}\in VG'-T''$; and (ii) $(x_i,e_i,e_{i+1},1,1)\in\Pi$ for $i=1,\dots,k-1$. If, in addition, $(x_i,e_i,e_{i+1},-1,-1)\in\Pi$ for $i=1,\dots,k-1$, P is called a $(\pm)$-T'-*line* (respectively, a $(\pm)$-T'-T''–line). If in the definition of a $(+)$-T'–line condition (i) is replaced by (i'): all $x_0,\dots,x_k$ are not in T'', $x_0=x_k$, and $(x_0,e_1,e_k,1,1)\in\Pi$, we say that P is a $(+)$-*circuit*. Similarly, if, in addition, $(x_i,e_i,e_{i+1},-1,-1)\in\Pi$ for $i=1,\dots,k$ (letting $e_{k+1}:=e_1$), P is called a $(\pm)$-*circuit*.

In further sections we use the following two statements of general nature that concern the case when no augmenting path exists.

Statement 3.6. (i) *If $P=(x_0,e_1,x_1,\dots,e_k,x_k)$ is a $(+)$-T'-line (resp. a $(+)$-T'-T''–*

line) then $\rho(P) \ge 2$ *(resp.* $\rho(P) \ge 1$*).*

(ii) *If* P *is a* $(\pm)$*-*T'*-line (resp. a* $(\pm)$*-*T'*-*T''*–line) then* $\rho(P) = 2$ *(resp.* $\rho(P) = 1$*).*

(iii) *If* P *is a* $(+)$*-circuit (resp. a* $(\pm)$*-circuit)) then* $\rho(P) \ge 0$ *(resp.* $\rho(P) = 0$*).*

Proof. We show that for any $v \in VG' - T''$ and $e, e' \in E_v$:

$$\text{(3.17)} \qquad \begin{aligned} \rho(v,e) + \rho(v,e') &\ge 0 \quad \text{if } (v,e,e',1,1) \in \Pi; \\ &\le 0 \quad \text{if } (v,e,e',-1,-1) \in \Pi. \end{aligned}$$

This implies the statement since, first, $k \ge 1$; second,

$$\rho(P) = \rho(x_0, e_1) + \rho(x_k, e_k) + \sum(\rho(x_i, e_i) + \rho(x_i, e_{i+1}) \ : \ i = 1, \ldots, k-1);$$

third, if $x_0 \in T'$ and $x_k \in T'$ (resp. $x_k \in T'' - T'$) then $\rho(x_0, e_1) = 1$ and $\rho(x_k, e_k) = 1$ (resp. $\rho(x_k, e_k) = 0$), by (3.14); and, fourth, (3.17) shows that for $i = 1, \ldots, k-1$ (resp. for $i = 1, \ldots, k$), $\rho(x_i, e_i) + \rho(x_i, e_{i+1})$ is nonnegative if $(x_i, e_i, e_{i+1}, 1, 1) \in \Pi$, and nonpositive if $(x_i, e_i, e_{i+1}, -1, -1) \in \Pi$ (letting $e_{k+1} := e_1$).

To see (3.17), consider two cases.

(i) v is unlabeled. Then $\rho(v, e) = \rho(v, e') = 0$, by (3.14).

(ii) $v \notin T''$ is labeled. Let $(Q = (y_0, u_1, y_1, \ldots, u_m, y_m), \Sigma = (\sigma_1, \ldots, \sigma_m))$ be an active path with $y_m = v$, and let $q = (y_{m-1}, y_m)$. Suppose that $\rho(v, e) + \rho(v, e') = \sigma + \sigma'$ is negative, where $\sigma := \sigma(v, e)$ and $\sigma' := \sigma(v, e')$. Then $\sigma = \sigma' = -1$. We observe that $(v, e, e', 1, 1) \notin \Pi$. For otherwise $(v, u_m, e, \sigma_m, \sigma = -1), (v, e, e', 1, 1) \in \Pi$ would imply $\tau := (v, u_m, e', \sigma_m, 1) \in \Pi$ (by (3.1)(iii)), and then $\tau \in \Pi$ and $(v, u_m, e', \sigma_m, \sigma' = -1) \in \Pi$ would imply $(v, u_m, u_m, \sigma_m, \sigma_m) \in \Pi$, whence there exists an augmenting path. Now suppose that $\sigma + \sigma' > 0$, i.e. $\sigma = \sigma' = 1$. Applying similar arguments we conclude that $(v, e, e', -1, -1) \notin \Pi$.

This proves (3.17) and the statement. •

Note that this proof shows that

(3.18) if $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is a $(+)$-T'-line or a $(+)$-T'-T''–line (respectively, a $(\pm)$-T'-line or a $(\pm)$-T'-T''–line) and $P_i := P\langle x_0, x_i\rangle$ then for $i = 1, \ldots, k-1$, $\rho(P_i) \ge 1 + \rho(x_i, e_i)$ (respectively, $\rho(P_i) = 1 + \rho(x_i, e_i)$).

Statement 3.7.

(i) *If* $\rho(e) > 0$ ($\rho(e) < 0$) *for some* $e = vz \in EG'$ *with* $v, z \notin T'' - T'$ *then* e *is unlabeled, at least one end of* e *is labeled, and* $h(e) = c''(e)$ *(resp.* $h(e) = c'(e)$*).*

(ii) *If* h *is perfect then all vertices in* H^h *are unlabeled; in particular,* $h(e)$ *is integral for all labeled edges* e.

Proof. (i) e is unlabeled by (3.16)(ii); and if both x, z were unlabeled, we would have $\rho(e) = 0$, by (3.14). Let v be labeled. Let $\rho(e) > 0$. Then $\rho(v, e) = 1$ (since $\rho(v, e) \neq 0$ and $\rho(z, e) \leq 1$). If $v \in T'$ then $h(e) = c''(e)$ (otherwise e would be labeled). Assume that $v \notin T'$, and let $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ be an active path with $x_k = v$. Since $\sigma(v, e) = \rho(v, e) = 1$, we have $(v, e_k, e, \sigma_k, 1) \in \Pi$, whence $h(e) = c''(e)$ (otherwise the path $((x_0, e_1, x_1, \ldots, e_k, x_k, e, z), (\sigma_1, \ldots, \sigma_k, 1))$ would be active and e would be labeled). Similarly, $\rho(v, e) < 0$ implies $h(e) = c'(e)$ (note that in this case $v \in T'$ is impossible).

(ii) If there is an active path that meets a circuit in H^h then there is an augmenting path (by arguments as in the proof of (3.9)). •

In the rest of this section we describe a natural approach to finding an augmenting path. Let us say that $e \in EG'$ is *positively* (*negatively*) *feasible* if $h(e) < c''(e)$ (resp. $h(e) > c'(e)$).

We grow, step by step, a digraph $D = (VD, AD)$ along with a mapping $\sigma : AD \to \{-1, 1\}$. For each arc $q = (x, y)$ of the current D there is a path $(x_0, q_1, x_1, \ldots, q_k, x_k)$ in D such that $q_i = (x_{i-1}, x_i)$ $(i = 1, \ldots, k)$, $q_k = q$, and $((x_0, e^{q_1}, x_1, \ldots, e^{q_k}, x_k), (\sigma^{q_1}, \ldots, \sigma^{q_k}))$ is an active path, where $e^{q'}$ is the edge of G' underlying $q' \in AD$. Initially, $D = (T, \emptyset)$. One may think that at each step we scan vertices v of the current D and scan edges e incident to v.

(i) If $e = vz$ is positively feasible, and $v \in T'$, we add to D the arc $q = (v, z)$ with $\sigma^q = 1$ (if such a pair $(q, 1)$ has not been added earlier).

(ii) If $e = vz$ is positively (negatively) feasible, and there is an arc $q' = (x, v) \in AD$ (possibly $x = z$) such that $(v, e^{q'}, e, \sigma^{q'}, 1) \in \Pi$ (resp. $(v, e^{q'}, e, \sigma^{q'}, -1) \in \Pi$), we add to D the arc $q = (v, z)$ with $\sigma^q = 1$ (resp. $\sigma^q = -1$) if such a pair (q, σ^q) has not been added earlier.

(iii) If we reach some $s \in T''$ by an arc $q = (v, s) \in AD$ with $\sigma^q = 1$ then there is an augmenting path.

(iv) If there appeared two pairs $(q = (x, y), 1)$ and $(q' = (x, y), -1)$ or two pairs $(q = (x, y), 1)$ and $(q' = (y, x), 1)$ such that $e^q = e^{q'}$ then there is an augmenting path (by arguments as in the proof of Lemma 3.4).

Clearly the process of growing (D, σ) can be fulfilled in strongly polynomial time, as well as extracting an augmenting path (P', Σ') from (D, σ) in cases (iii)-(iv). Moreover, if h is perfect (semi-perfect), we can modify the (P', Σ') to get an augmenting path (P, Σ) as in (3.9) (resp. (3.10)). When the current (D, σ) can be no longer enlarged by use of the operations in (i) or (ii) and the "break-through" as in (iii)-(iv) does not occur, this (D, σ) satisfies (3.13).

4. Pseudo-polynomial algorithm

In this section we combine results of Sections 2 and 3 to design a pseudo-polynomial algorithm for solving (1.2). This facilitates the construction of more efficient algorithms developed in the following sections. We may assume that $c(e) > 0$ for all $e \in EG$ (as if $c(e) = 0$, we can delete e from G), and that there is at least one path in G connecting different terminals (for otherwise zero multiflow is optimal).

To apply results of Section 2, it is important to assume that the cost function a is positive. Such an assumption does not lead to loss of generality. Indeed, if the set $Z := \{e \in EG : a(e) = 0\}$ is nonempty, we may consider, instead of a, the perturbed function a' defined by

$$\text{(4.1)} \qquad \begin{aligned} a'(e) &:= (2c(Z) + 1)a(e) \qquad \text{for } \ e \in EG - Z, \\ &:= 1 \qquad \text{for } \ e \in Z. \end{aligned}$$

Then for any two *half-integral* maximum multiflows f and f' with $\Delta := a_{f'} - a_f > 0$ we have $\Delta \geq 1/2$ and

$$\begin{aligned} a'_{f'} - a'_f &= (2c(Z) + 1)(a_{f'} - a_f) + (\zeta^{f'}(Z) - \zeta^f(Z)) \\ &\geq (2c(Z) + 1)\Delta - c(Z) \geq c(Z) + \frac{1}{2} - c(Z) > 0. \end{aligned}$$

Therefore, if f is an optimal (half-integral) solution of (1.2) for a' then f is an optimal solution of (1.2) for a as well.

To solve (1.2), we, in fact, consecutively construct optimal solutions to (1.3) and its dual for a sequence $p_0 < p_1 < \ldots < p_M$ of values of parameter p of (1.3). More precisely, at a current iteration we deal with some p, γ, l, Γ, h, where

$$\text{(4.2)} \quad \gamma \in \mathbb{Q}_+^{EG}; \quad l := a + \gamma; \quad p := p^l \ \text{ (cf. (2.5))}; \quad \Gamma = \Gamma^l \ \text{ is defined in Section 2};$$

(4.3) $\qquad h$ is a regular function on $E\Gamma$;

(4.4) $\gamma(e)=0$ for all $e\in EG-E\Gamma$; $\quad$ for $e\in E\Gamma$, if $\gamma(e)>0$ then $h(e)=c(e)$.

Note that l is positive since a is positive and γ is nonnegative. Clearly, a multiflow f obtained from h by a decomposition as in Statement 2.3 and the function γ satisfy (2.3)-(2.4), i.e., they are optimal primal and dual solutions to (1.3) with p determined by γ as in (4.2); we say that h and γ satisfying (4.3)-(4.4) are *optimal.*

We start with $\gamma:=0$ and $h:=0$, which are obviously optimal. At a current iteration we either replace h by a new regular function h' so that $v_{h'}>v_h$ and h',γ are optimal, or replace γ by a new γ' so that $p^{l'}>p$ (for $l':=a+\gamma'$) and h,γ' are optimal. This method is, in essence, within frameworks of the primal-dual l.p. method and is in accordance with the classical approach, due to Ford and Fulkerson [FF], to solve the minimum-cost maximum-flow problem. Of course, for the problem in question we need more involved combinatorial tools to realize such a method.

To do this, we form the following transitive fork environment Π for $G':=\Gamma$, $T':=T'':=T$ and h, using the same notation as in Section 2:

(4.5) for $v\in V\Gamma-T$, $e,e'\in E(v)$ and $\sigma,\sigma'\in\{-1,1\}$, the tuple $\tau=(v,e,e',\sigma,\sigma')$ is in Π if and only if there is no $s\in\langle T\rangle$ such that s is tight for v,h, and: either

- (i) $e,e'\in E_s(v)$ and $\sigma,\sigma'=1$; or
- (ii) $e,e'\notin E_s(v)$ and $\sigma,\sigma'=-1$; or
- (iii) $e\in E_s(v)$, $e'\notin E_s(v)$, $\sigma=1$ and $\sigma'=-1$; or
- (iv) $e\notin E_s(v)$, $e'\in E_s(v)$, $\sigma=-1$ and $\sigma'=1$.

If there is a tight $s\in\langle T\rangle$ such that some of (i)-(iv) holds, we say that s *forbids* the tuple (v,e,e',σ,σ').

Statement 4.1. Π *satisfies (3.1).*

Proof. (3.1)(i) is obvious. To see (3.1)(ii), consider $v\in V\Gamma-T$, $e,e'\in E(v)$ and $\sigma\in S(e)=S(e,h)$. Suppose that neither $\tau=(v,e,e',\sigma,1)$ nor $\tau'=(v,e,e',\sigma,-1)$ is in Π. Let s forbid τ and s' forbid τ'. If $\sigma=1$ then e is in both $E_s(v)$ and $E_{s'}(v)$, whence $s=s'$. Now $e'\in E_s(v)$ (since s forbids τ) and $e'\notin E_s(v)$ (since s forbids τ'); a contradiction. And if $\sigma=-1$ then $e\notin E_s(v)\ni e'$ and $e,e'\notin E_{s'}(v)$. Hence, $s\neq s'$ and $E_s(v)\cap E_{s'}(v)=\emptyset$, whence

$$h(E(v))\geq h(E_s(v))+h(E_{s'}(v))+h(e)=\frac{1}{2}h(E(v))+\frac{1}{2}h(E(v))+h(e).$$

So $h(e)=0$. Then $S(e)=\{1\}$, contrary to the fact that $\sigma=-1$. Thus, (3.1)(ii) is true.

Suppose that $\tau=(v,e,e',\sigma,\sigma')$ and $\tau'=(v,e',e'',-\sigma',\sigma'')$ but $\tau''=(v,e,e'',\sigma,\sigma'')$ are in Π. Without loss of generality let $\sigma'=1$, and let s forbid τ''. It is easy to see that if $e'\in E_s(v)$ then s forbids τ, while if $e'\notin E_s(v)$ then s forbids τ'; a contradiction. This proves (3.1)(iii).

Finally, consider $\tau_1=(v,e_1,e_2,\sigma_1,\sigma_2)$ and $\tau_2=(v,e_2,e_3,\sigma_2,\sigma_3)$ as in (3.1)(iv); we show that $\tau_3=(v,e_1,e_3,\sigma_1,\sigma_3)\notin\Pi$. For $i=1,2$ let $s_i\in\langle T\rangle$ be a tight element that forbids τ_i.

(a) Let $\sigma_2=1$. Then $e_2\in E_{s_i}(v)$ for $i=1,2$, whence $s_1=s_2=:s$. Furthermore, for $i=1,3$, $e_i\in E_s(v)$ if and only if $\sigma_i=1$. This implies that s forbids τ_3.

(b) Let $\sigma_2=-1$. If $s_1=s_2=:s$ then, by the argument as above, s forbids τ_3. So assume that $s_1\neq s_2$. Then $E_{s_1}(v)\cap E_{s_2}(v)=\emptyset$, $e_2\notin E_{s_i}$ $(i=1,2)$, and $h(e_2)>0$ (as $-1=\sigma_2\in S(e_2)$). We get a contradiction since

$$h(E(v))\geq h(E_{s_1}(v))+h(E_{s_2}(v))+h(e_2)>\frac{1}{2}h(E(v))+\frac{1}{2}h(E(v))=h(E(v)). \quad \bullet$$

Thus, we can apply results of Section 3 to the current Γ,T,h,Π,c',c'', where for $e\in E\Gamma$,

$$\begin{aligned} &c'(e):=0 \text{ and } c''(e):=c(e) \quad \text{if } \gamma(e)=0,\\ &c'(e)=c''(e):=c(e) \quad \text{if } \gamma(e)>0. \end{aligned} \tag{4.6}$$

We assume that the current h is perfect (see (3.8)(ii)). As before, the subgraph induced by the edges e with non-integral $h(e)$ is denoted by $H=H^h$.

We grow the digraph D along with the mapping σ as explained in the end of Section 3. Note that an edge $e\in E\Gamma$ is (positively or negatively) feasible only if $\gamma(e)=0$ (by (4.6)), so each edge e' with $\gamma(e')>0$ is unlabeled. Suppose that the process of growing D results in finding an augmenting path $(P=(x_0,e_1,x_1,\dots,e_k,x_k),\Sigma=(\sigma_1,\dots,\sigma_k))$. We may assume that (P,Σ) satisfies (3.9). Note that (P,Σ) as in (3.9) obviously satisfies (3.10). We establish the following result which will be also used in Sections 5-6.

Statement 4.2. *Let h be inner half-Eulerian. Let $(P=(x_0,e_1,x_1,\dots,e_k,x_k),\Sigma=(\sigma_1,\dots,\sigma_k))$ be an augmenting path satisfying (3.10). Let h' be obtained from h and (P,Σ) as in (3.12). Then h' is c-admissible and regular.*

Proof. c-admissibility of h' immediately follows from (3.3)(i), (3.10)(iv) and (3.12). To show that h' is regular, consider $v \in V\Gamma - T$ and $s \in \langle T \rangle$. Let

(4.7) $b := h(E_s(v)), \quad \overline{b} := h(E(v) - E_s(v)), \quad b' := h'(E_s(v)), \quad \overline{b}' := h'(E(v) - E_s(v)).$

Since h is inner half-Eulerian and regular, $\Delta := \overline{b} - b$ is a nonnegative integer. We have to show that $\Delta' := \overline{b}' - b'$ is nonnegative. If v is not in P then $b' = b$ and $\overline{b}' = \overline{b}$. So we assume that $v \in P$. Three cases are possible.

Case 1. $v = x_i$ for exactly one i, and we are not in (3.12)(i). Let $e := e_i$, $e' := e_{i+1}$, $\sigma := \sigma_i$, $\sigma' := \sigma_{i+1}$. Then $h'(e) = h(e) + \sigma/2$ and $h'(e') = h(e') + \sigma'/2$ if $e \neq e'$, and $h'(e) = h(e) + \sigma$ if $e = e'$ (as $\sigma = \sigma'$). Therefore, $|b' - b| + |\overline{b} - \overline{b}'| \leq 1$, whence $\Delta \geq 1$ implies $\Delta' \geq 0$. And if $\Delta = 0$ (i.e., s is tight for v, h) then the fact that $\tau = (v, e, e', \sigma, \sigma')$ is not forbidden by s (as τ is a fork) easily implies that $b' - b \leq \overline{b}' - \overline{b}$, whence $\Delta' \geq \Delta = 0$.

Case 2. $v = x_i = x_j$ for $i \neq j$. For simplicity we assume that all $e := e_i$, $e' := e_{i+1}$, $u := e_j$, $u' := e_{j+1}$ are different (if some of these edges coincide, arguments are similar). Then $h'(e_r) = h(e_r) + \sigma_r/2$ for $r = i, i+1, j, j+1$, and $h'(e'') = h(e'')$ for the other edges e'' in $E(v)$.

Since $(v, e, u', \sigma_i, \sigma_{j+1}) \notin \Pi$ (by (3.10)(i)), there is $s' \in \langle T \rangle$ tight for h, v, i.e. $q = \overline{q}$ for $q := h(E_{s'}(v))$ and $\overline{q} := h(E(v) - E_{s'}(v))$. Moreover, $\sigma_i = 1$ implies that $e \in E_{s'}(v)$, while $\sigma = -1$ implies that $e \notin E_{s'}(v)$. Then the fact that $\tau = (v, e, e', \sigma_i, \sigma_{i+1})$ is not forbidden by s' (as $\tau \in \Pi$) shows that $e' \in E_{s'}(v)$ if $\sigma_{i+1} = -1$ and $e' \notin E_{s'}(v)$ if $\sigma_{i+1} = 1$. Similarly, $u' \in E_{s'}(v)$ if and only if $\sigma_{j+1} = 1$, whence $u \in E_{s'}(v)$ if and only if $\sigma_j = -1$. From these arguments it follows that $q' - q = \overline{q}' - \overline{q}$, where $q' := h'(E_{s'}(v))$ and $\overline{q}' := h'(E(v) - E_{s'}(v))$. Thus, $q' - \overline{q}' = q - \overline{q} = 0$, i.e., s' is tight for h', v.

Now for $s, b', \overline{b}'$ as above we have $b' = q'$ and $\overline{b}' = \overline{q}'$ if $s = s'$, and $b' \leq \overline{q}'$ and $\overline{b}' \geq q'$ if $s \neq s'$ (as $E_s(v) \cap E_{s'}(v) = \emptyset$). Hence, $\Delta' \geq 0$.

Case 3. $v = x_i$ and case (3.12)(i) occurs. Apply to $e := u' := e_i$ and $e' := u := e_{i+1}$ arguments similar to those in Case 2. •

So we make the transformation $h \xrightarrow{P,\Sigma} h'$ as in (3.12), which results in a half-Eulerian, c-admissible, and regular h' with $v_{h'} > v_h$. Furthermore, h' satisfies (4.4). Thus, the pair (h', γ) is optimal.

Note that each circuit in $H^{h'}$ is a circuit in H^h (and it remains non-eliminatable) except, possibly, one circuit C. Such a C can appear only in case (iii) of (3.9), then C coincides with $P\langle x_m, x_{k-m} \rangle$. We examine whether C is eliminatable with respect to h' (and the corresponding set Π' of forks). If not, the function h' is perfect. Otherwise we apply to C the elimination procedure (C) from Section 3, by making the transformation

$h' \xrightarrow{C,\Sigma'} h''$ (with the corresponding Σ'). The resulting h'' is regular, $v_{h''} = v_{h'}$, and (4.4) holds for h'', γ, i.e. h'' is a perfect optimal function (the regularity of h'' is shown by use of arguments as in Case 2 of the proof of Statement 4.2).

Now we consider the case when no augmenting path exists. By Lemma 3.2 the resulting (D, σ) satisfies (3.13). We transform γ to γ' using the numbers $\rho(e)$, $e \in E\Gamma$, defined in (3.15). First of all we observe the following.

Statement 4.3. *Let $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ be a path in Γ connecting different terminals x_0 and x_k. Let $i \in \{1, \dots, k-1\}$, $v := x_i$, $e := e_i$ and $e' := e_{i+1}$.*

(i) *If P is a T-line then $\tau = (v, e, e', 1, 1) \in \Pi$.*

(ii) *If there is a decomposition $\mathcal{D} = \{(P_j, \lambda_j) : i = 1, \dots, m\}$ of h (see Statement 2.3) with $P_1 = P$ then $\tau' = (v, e, e', -1, -1) \in \Pi$.*

Proof. Let $s := s(v, e)$ and $s' := s(v, e')$. (i) immediately follows from (4.5) and the fact that $s \ne s'$ (cf. (2.8)). To see (ii), assume that $P_1, \dots P_r$ are the paths in $\mathcal{D}$ passing through v. It is easy to see that for any $q \in \langle T \rangle$ tight for v, each P_j $(1 \le j \le r)$ has exactly one edge in $E_q(v)$ (taking into account (2.8) and the fact that all λ_j's are positive); in particular, $|\{e, e'\} \cap E_q(v)| = 1$. This implies that $\tau' \in \Pi$. $\bullet$

Let us say that a T-path is a *strong line* if it is a member of some decomposition of h. Statement 4.3 shows that a T-line is a $(+)$-T-line and that a strong line is a $(\pm)$-T-line, cf. Definition 3.5. Statements 3.6 and 3.7 lead to the following important statement concerning transformation of the dual solution γ.

Statement 4.4. *For $\varepsilon \in \mathbb{R}_+$ let γ^ε be the function on EG defined by*

$$
\begin{aligned}
(4.8) \qquad \gamma^\varepsilon(e) &:= \gamma(e) + \varepsilon\rho(e) \qquad \text{for } e \in E\Gamma \text{ with } h(e) = c(e), \\
&:= 0 \ \ (= \gamma(e)) \qquad \text{for the other } e \in EG.
\end{aligned}
$$

There exists $\varepsilon > 0$ such that for any $0 \le \varepsilon' \le \varepsilon$:

(i) *$\gamma^{\varepsilon'}$ is nonnegative;*

(ii) *$p^{l'}$ is $p + 2\varepsilon'$, where $l' := l^{\varepsilon'} := a + \gamma^{\varepsilon'}$.*

Proof. Set

$$(4.9) \qquad E^- := \{e \in E\Gamma : h(e) = c(e) \text{ and } \rho(e) < 0\}, \qquad \text{and}$$

$$(4.10) \qquad \varepsilon_1 := \min\{-\gamma(e)/\rho(e) : e \in E^-\}.$$

By (4.8), $\gamma^{\varepsilon'}(e) \ge \gamma(e) \ge 0$ for any $e \in EG - E^-$. Let $e = xy \in E^-$. Since $\rho(e) < 0$, $h(e) = c'(e)$, by Statement 3.7(i). We know that $c(e) > 0$. Thus, $c'(e) = h(e) = c(e) > 0$, whence $\gamma(e) > 0$, by (4.6). This shows that $\varepsilon_1 > 0$ and that (i) is true when $\varepsilon = \varepsilon_1$.

To see (ii), first note that Γ contains at least one T-line P' (since some terminals in G are connected by a path); moreover, $c(e) > 0$ holds for each $e \in P'$. Therefore, $h \ne 0$ (otherwise P' would be augmenting). This implies that there exists a strong line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$. Then P is a $(\pm)$-line; whence $\rho(P) = 2$ (by Statement 3.6(ii)). We observe that $\gamma^{\varepsilon}(e_i) = \gamma(e_i) + \varepsilon\rho(e_i)$ for $i = 1, \dots, k$. This follows from (4.8) if $h(e) = c(e)$. And if $h(e) < c(e)$ then we have $c'(e) = 0 < h(e) < c(e) = c''(e)$, whence $\rho(e_i) = 0$ (by Statement 3.7(i)). Now, for any $\varepsilon' \in \mathbb{R}_+$ and $l' := a + \gamma^{\varepsilon'}$,

$$l'(P) = a(P) + \gamma^{\varepsilon'}(P) = a(P) + \gamma(P) + \varepsilon'\rho(P) = l(P) + 2\varepsilon' = p + 2\varepsilon'.$$

Next, let $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ be a T-line. Then P is a $(+)$-line, and $\rho(P) \ge 2$ (by Statement 3.6(i)). We observe that $\gamma^{\varepsilon}(e_i) \ge \gamma(e_i) + \varepsilon\rho(e_i)$ for $i = 1, \dots, k$. Indeed, this is true if $h(e_i) = c(e_i)$, by (4.8). And if $h(e_i) < c(e_i)$ $(= c''(e_i))$ then $\rho(e_i) \le 0$ (Statement 3.7(i)), and we have $\gamma^{\varepsilon}(e_i) = 0 = \gamma(e_i) \ge \gamma(e_i) + \varepsilon\rho(e_i)$. Thus, $l'(P) \ge p + 2\varepsilon'$.

Finally, let P be a simple T-path in G that is not a T-line, i.e., $l(P) > p$. Since $l^{\varepsilon'}(P)$ is a continuous function of ε', and $l = l^{\varepsilon'}$ for $\varepsilon' = 0$, there is $\varepsilon > 0$ such that $l'(P) \ge p + 2\varepsilon'$ for all $0 \le \varepsilon' \le \varepsilon$. Now (ii) follows from finiteness of the set of simple T-paths. •

Set

$$\widetilde{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2\}, \tag{4.11}$$

where ε_1 is defined in (4.10), whereas ε_2 is the maximum $\overline{\varepsilon} \le \varepsilon_1$ such that for $0 \le \varepsilon' < \overline{\varepsilon}$, $l^{\varepsilon'}(P) \ge p + 2\varepsilon'$ holds for any (simple) T-path P in G. By Statement 4.4, $\widetilde{\varepsilon} > 0$. Assuming that $\widetilde{\varepsilon} < \infty$, let $\gamma' := \gamma^{\widetilde{\varepsilon}}$, $l' := a + \gamma'$ and $p' := p + 2\widetilde{\varepsilon}$. Then γ' is just the function to which γ is transformed. We assert that the pair (h, γ') is optimal. (4.4) follows from (4.8). To see (4.3), we observe from the proof of Statement 4.4 that

(4.12) $l'(P_i) = p'$ holds for each path in a decomposition $\mathcal{D} = \{(P_i, \lambda) : i = 1, \dots, m\}$ of h, i.e., P_i is a T-line for l'.

In particular, $h(e) > 0$ implies that e is in $\Gamma' := \Gamma^{l'}$. We may assume that h is extended by zero to $E\Gamma' - E\Gamma$. Since h coincides with the function ζ^f for a multiflow

f going along T-lines in Γ', h is regular in Γ' (see (2.9)). Thus, (h, γ') is optimal, as required.

When $\widetilde{\varepsilon} = \infty$, h gives an optimal solution to (1.2) since for any (in particular, rather large) $\varepsilon' \geq 0$, the pair $(h, \gamma^{\varepsilon'})$ is optimal for $p^{l'} = p + 2\varepsilon'$ (where $l' := a + \gamma^{\varepsilon'}$).

Remark 4.5. It can be directly proved that if $\widetilde{\varepsilon} = \infty$ then the current h has the maximum possible value v_h (or, by (2.13), that a multiflow f obtained from a decomposition of h is maximum). By a theorem due to Lovász [Lo] and, independently, to Cherkassky [Ch], the maximum value of a c-admissible multiflow for G, T, h is equal to

$$\frac{1}{2} \sum (c(\delta(X_s)) \,:\, s \in T), \tag{4.13}$$

where each $\delta(X_s)$ is a minimum capacity cut in G separating $s \in T$ from $T - \{s\}$ (i.e., $|X_s \cap T| = s$). (For $X \subseteq VG$, $\delta(X)$ is the set of edges connecting X and $VG - X$.) To see this equality, take a large ε', and define X_s to be the set of vertices in Γ' reachable from s by active paths. The sets X_s are mutually disjoint. For if $X_s \cap X_t \neq \emptyset$ for some $s \neq t$ then s and t are connected in Γ by a path P all edges of which are labeled. By (3.16), $\rho(e) = 0$ for all edges $e \in P$, so $l'(e) = l(e)$ for any ε', whence $p^{l'}$ is bounded by $l(P)$. We leave it to the reader to check that for each $s \in T$ the edges of $\delta(X_s)$ are saturated by h and, moreover, $c(\delta(X_s)) = h(E(s))$, whence v_h is equal to the value in (4.13).

The above arguments naturally provide an algorithm to solve (1.2). We say that the current iteration is *primally increasing* if h is transformed into a new h', and *dually increasing* if γ is transformed into a new γ'. Since $v_{h'} \geq v_h + 1/2$, the number of primally increasing iterations is $O(c(EG))$. To conclude that the algorithm is finite, we show the following.

Lemma 4.6. *The number of consecutive dually increasing iterations is* $O(|EG|)$.

This lemma will follow from Statements 4.7-4.11 below. Consider a dually increasing iteration, let $h, \gamma, l, p, \Gamma, \Pi, (D, \sigma), \rho, \widetilde{\varepsilon} < \infty$ denote corresponding objects at this iteration, and let $\gamma', l', p', \Gamma', \Pi', (D', \sigma')$ be the corresponding objects at the next iteration. A T-line P in Γ is called *non-broken* if $l'(P) = p'$ $(= p + 2\widetilde{\varepsilon})$; so P is a T-line in Γ'. In particular, every strong line is non-broken, whence each edge $e \in E\Gamma$ with $h(e) > 0$ belongs to a non-broken T-line. Also from Statement 3.6 and the fact that $\gamma'(e) \geq \gamma(e) + \widetilde{\varepsilon}\rho(e)$ for any $e \in E\Gamma$ (see arguments in the proof of (ii) in Statement 4.4) we conclude that

(4.14) a T-line $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ in Γ is non-broken if and only if $\gamma'(e_i) = \gamma(e_i) + \rho(e_i)$ for $i = 1, \dots, k$, and $\rho(x_i, e_i) + \rho(x_i, e_{i+1}) = 0$ for $i = 1, \dots, k-1$.

In what follows a part of a T-line in Γ is called a *line*. The concatenation of paths $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ and $Q = (y_0, u_1, y_1, \dots, u_m, y_m)$ with $x_k = y_0$ is denoted by $P \cdot Q$.

Statement 4.7. *Each labeled edge $e \in E\Gamma$ belongs to a non-broken line.*

Proof. Let $(P = (x_0, e_1, x_1, \dots, e_k, x_k), \Sigma = (\sigma_1, \dots, \sigma_k))$ be an active path with $e = e_j$ for some j. We may assume that $h(e) = 0$, whence $\sigma_k = 1$. Also we may assume that (P, Σ) is chosen so that $\sigma_i = 1$ for $i = j, \dots, k$, and $k - j$ is as large as possible. Let r be the minimal index such that $\sigma_i = 1$ for $i = r, \dots, j$. We observe that $P' = P\langle x_r, x_k\rangle$ is a line (in particular, $k - r$ is finite). This follows from (2.8) and the fact that $s(x_i, e_i) \neq s(x_i, e_{i+1})$ for $i = r+1, \dots, k-1$. The latter property is true because if $e_i, e_{i+1} \in E_s(x_i)$ for some $s \in \langle T\rangle$ then s is not tight for x_i (otherwise $(x_i, e_i, e_{i+1}, 1, 1)$ is not a fork); therefore $(x_i, e_i, e_i, 1, 1) \in \Pi$, whence the concatenation of $(P, \Sigma)\langle x_0, x_i\rangle$ and its reverse is augmenting.

Put $x := x_r$, $e' := e_r$ and $w := e_{r+1}$. We form a line Q as follows. If $x \in T$ (i.e., $r = 0$), put $Q := P'$. Let $r \geq 1$. Then $\sigma_r = -1$ (by the minimality of r). Hence $h(e') > 0$, so e' belongs to a strong line L, say, from $\widetilde{s}$ to $\widetilde{t}$. We put Q to be the concatenation of the path L' reverse to $L\langle x, \widetilde{t}\rangle$ and P'. We assert that such a Q is a line. To see this, it is suffices to check that $s(x, w) \neq s(x, e'')$, where e'' is the last edge in L'. Suppose that $s(x, w) = s(x, e'')$. Since $(x, e', e', -1, -1) \notin \Pi$ (otherwise there is an augmenting path), $e' \notin E_{s'}(x)$ for some $s \in \langle T\rangle$ tight for x. Note that $(x, e', e'', -1, -1) \in \Pi$ (since L is strong), whence $e'' \in E_s(x)$. Therefore, $w \in E_s(x)$. Then s forbids $(x, e', w, \sigma_r, \sigma_{r+1})$; a contradiction.

Thus, in both cases Q is a line beginning at a terminal. We observe that

$$l'(Q) = l(Q). \tag{4.15}$$

This follows from (3.16) if $Q = P'$. And if $Q = L' \cdot P'$ then $l'(P') = l(P')$, and $l'(L') = l(L')$. The latter equality follows from (3.18) and the fact that $\rho(x, e'') = -\rho(x, e') = -1$ (since $\sigma_r = -1$ implies $\rho(x, e') = 1$, by (3.16)(ii)).

Now consider a T-line R, say, from $\widetilde{t}$ to v, extending Q; let $R = Q \cdot R'$. Let u be the edge in R following $y := x_k$. The minimality of $k - j$ shows that $h(u) = c(u)$ (since $s(y, e_k) \neq s(y, u)$ implies that $(y, e_k, u, 1, 1) \in \Pi$). Hence, u belongs to a strong line, and without loss of generality we may assume that R' is a part of a strong line. By (3.18) and (4.8), $l'(R') \le l(R') + \widetilde{\varepsilon}(1+\rho(y,u)) \le l(R') + 2\widetilde{\varepsilon}$. Hence, $l'(R) = l'(Q) + l'(R') \le l(Q) + l(R') + 2\widetilde{\varepsilon} \le l(R) + 2\widetilde{\varepsilon}$. This means that R is a non-broken T-line containing e. •

Statement 4.8. *Let $(P = (x_0, e_1, x_1, \dots, e_k, x_k), \Sigma = (\sigma_1, \dots, \sigma_k))$ be an active path in Γ. Then (P, Σ) is active in Γ'.*

Proof. If $k \le 1$, the statement is trivial. So we may assume by induction that $k > 1$ and $(P,\Sigma)\langle x_0, x_{k-1}\rangle$ is active in Γ'. Let $x := x_{k-1}$, $e := e_{k-1}$, $e' := e_k$, $\sigma := \sigma_{k-1}$, $\sigma' := \sigma_k$. By Statement 4.7, there are non-broken T-lines Q and Q' containing e and e', respectively. Let $Q = Q_1 \cdot Q_2$ and $Q' = Q'_1 \cdot Q'_2$, where e and x are the last edge and vertex in Q_1, and e' and x are the last edge and vertex in Q'_1. It is easy to see that at least one of the following is true:

(4.16) (i) $L := Q_1 \cdot (Q'_1)^{-1}$ and $L' := Q_2^{-1} \cdot Q'_2$ are T-lines in Γ; or

(ii) $R := Q_1 \cdot Q'_2$ and $R' := Q'_1 \cdot Q_2$ are T-lines in Γ.

Note that if L and L' are T-lines in Γ then

$$2p' = l'(Q) + l'(Q') = l'(L) + l'(L')$$

together with $l'(L) \ge p'$ and $l'(L') \ge p'$ implies that $l'(L) = l'(L') = p'$, i.e. L and L' are non-broken. Similarly, if (4.16)(ii) occurs, R and R' are non-broken.

We may assume that if $\sigma = -1$ then Q is a strong line (as $h(e) > 0$); and similarly, if $\sigma' = -1$, Q' is a strong line. Using (3.16)(i),(3.17),(3.18) and the fact that $\gamma'(e) \ge \gamma(e) + \widetilde{\varepsilon}\rho(e)$ for any $e \in E\Gamma$, we observe that

(4.17) (i) if $\sigma = 1$ then $l'(Q_1) \ge l(Q_1)$ and $l'(Q_2) \ge l(Q_2) + 2\widetilde{\varepsilon}$;

(ii) if $\sigma = -1$ then $l'(Q_1) \ge l(Q_1) + 2\widetilde{\varepsilon}$ and $l'(Q_2) \ge l(Q_2)$;

(iii) if $\sigma' = 1$ then $l'(Q'_1) \ge l(Q'_1) + 2\widetilde{\varepsilon}$ and $l'(Q'_2) \ge l(Q'_2)$;

(iv) if $\sigma' = -1$ then $l'(Q'_1) \ge l(Q'_1)$ and $l'(Q'_2) \ge l(Q'_2) + 2\widetilde{\varepsilon}$.

Let $\tau = (x, e, e', \sigma, \sigma')$. Consider four possible cases.

Case 1. $\sigma = \sigma' = -1$. Then $l'(Q_1) + l'(Q'_2) \ge l(Q_1) + l(Q'_2) + 4\widetilde{\varepsilon}$ (by (4.17)(ii),(iv)), whence R cannot be a T-line in Γ'. Thus, L, L' are non-broken T-lines. Since Q and Q' are strong lines, there is a decomposition $\mathcal{D}$ of h containing pairs (Q,λ) and (Q',λ') (with positive λ, λ'); let for definiteness $\lambda \ge \lambda'$. Replacing these pairs by

$(Q, \lambda - \lambda'), (L, \lambda'), (L', \lambda')$, we obtain a decomposition $\mathcal{D}'$ of h. Hence, L is a strong line, and now Statement 4.3(ii) (applied to L and Γ') yields that $\tau \in \Pi'$.

Case 2. $\sigma = \sigma' = 1$. Then $l'(Q_1') + l'(Q_2) \geq l(Q_1') + l(Q_2) + 4\widetilde{\varepsilon}$ (by (4.17)(i),(iii)), whence R' cannot be a T-line in Γ'. Thus, L is non-broken. By Statement 4.3(i) (applied to L and Γ'), $\tau \in \Pi'$.

Case 3. $\sigma = -1, \sigma' = 1$. Then $l'(Q_1) + l'(Q_1') \geq l(Q_1) + l(Q_1') + 4\widetilde{\varepsilon}$ implies that R cannot be a T-line in Γ'. Hence, R, R' are non-broken. Let e'' be the first edge in Q_2. Since e', e'' belong to a T-line (R') in Γ', $\tau' = (x, e', e'', 1, 1) \in \Pi'$ (by Statement 4.3(i)). Since e, e'' belong to a strong line (Q) in Γ', $\tau'' = (x, e, e'', -1, -1) \in \Pi'$ (by Statement 4.3(ii)). Applying (3.1)(iii) to τ' and τ'', we obtain $\tau \in \Pi$.

Case 4. $\sigma = 1, \sigma' = -1$. Then $l'(Q_2) + l'(Q_2') \geq l(Q_2) + l(Q_2') + 4\widetilde{\varepsilon}$ implies that L' cannot be a T-line in Γ'. Hence, R, R' are non-broken. Let e'' be the first edge in Q_2'. We have $\tau' = (x, e, e'', 1, 1) \in \Pi'$ (as e, e'' are in a T-line (R) in Γ') and $\tau'' = (x, e', e'', -1, -1) \in \Pi'$ (as e', e'' are in a strong line (Q') in Γ'), whence $\tau \in \Pi'$. •

Statement 4.8 shows that

(4.18) D is a subgraph of D', and $(\sigma')^q = \sigma^q$ for all $q \in AD$.

Statement 4.9. *Suppose that there is an edge $u = xy \in E\Gamma$ with $\gamma(u) > 0$ and $\gamma'(u) = 0$. Then u is labeled in Γ'.*

Proof. Since $\gamma(u) > 0$, u is an unlabeled edge in Γ. Since $\gamma(u) > \gamma'(u)$, some end of u, x say, is in D', and hence $x \in VD'$. (4.8) shows that $\rho(u) < 0$, whence $x \notin T$ and $\rho(x, u) = \sigma(x, u, q) = -1$, where q is an arc in D entering x. Let $e = e^q$ and $\sigma = \sigma^q$; then $\tau = (x, e, u, \sigma, -1) \in \Pi$. Note that $\gamma(u) > 0$ implies that $h(u) = c(u) > 0$; so u belongs to a strong line P. Let Q be a non-broken T-line in Γ that contains e. Applying to Q and P arguments similar to those in the proof of Statement 4.8 (for Cases (i) and (iv)), we deduce that $\tau \in \Pi'$. Now the fact that u is negatively feasible in Γ' (as $\gamma'(u) = 0$ and $h(u) > 0$) implies that u must be labeled in Γ'. •

Thus, if $\widetilde{\varepsilon} = \varepsilon_1$ (see (4.11)) then AD' strictly includes AD. Now we consider the case when $\widetilde{\varepsilon} = \varepsilon_2$.

Statement 4.10. *There exists a T-path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ such that:* (i) *$l(P) > p$, $l'(P) = p'$, and* (ii) *there are $0 \leq i \leq j \leq k$ for which $P_1 := P\langle x_0, x_i\rangle$ and $P_2 := P\langle x_j, x_k\rangle$ are parts of non-broken T-lines, and each edge $e_{i+1}, \dots, e_j$ does not* *belongs to any non-broken line.*

Proof. Since $\widetilde{\varepsilon} = \varepsilon_2$, there exists a T-path $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ with $l(P) > p$ and $l'(P) = p'$. Let i be the maximum index such that $P\langle x_0, x_i\rangle$ is a part of a non-broken T-line, and let $j \ge i$ be the minimum index such that $P\langle x_j, x_k\rangle$ is a part of a non-broken T-line. Clearly $i \ne k$ and $j \ne 0$. Assuming that P is chosen so that $j - i$ is minimum, we assert that P satisfies (ii) of the statement.

Suppose, for a contradiction, that some $e := e_r$ $(i < r \le j)$ belongs to a non-broken T-line $Q = (y_0, u_1, y_1, \dots, u_m, y_m)$; let for definiteness $e = u_d$ and $x_r = y_d$. Form the paths $L_1 := P\langle x_0, x_r\rangle \cdot Q\langle y_r, y_m\rangle$ and $L_2 := Q\langle y_0, y_d\rangle \cdot P\langle x_r, x_k\rangle$. Since $l'(e) > 0$ and both $P\langle x_0, x_r\rangle$ and $Q\langle y_{d-1}, y_m\rangle$ are shortest for l', the case $x_0 = y_m$ is impossible. Similarly, $y_0 = x_k$ is impossible. Hence, L_1 and L_2 are T-paths. Moreover,

$$2p' = l'(P) + l'(Q) = l'(L_1) + l'(L_2)$$

implies that $l'(L_i) = p'$, $i = 1, 2$. Since $l(L_1) + l(L_2) = l(P) + l(Q) > 2p$, we observe that $l(L_i) > p$ for some i; let for definiteness $i = 1$. Then the parts $P\langle x_0, x_i\rangle$ and $Q\langle y_{d-1} = x_{r-1}, y_m\rangle$ of L_1 belong to non-broken T-lines, and $r - 1 - i < j - i$, contrary to the choice of P. $\bullet$

Let P, i, j, P_1, P_2 be as in Statement 4.10. By (4.8), $\Delta_r := (l'(P_r) - l(P_r)/\widetilde{\varepsilon}$ $(= (\gamma'(P_r) - \gamma(P_r))/\widetilde{\varepsilon})$ is an integer, $r = 1, 2$. Moreover, $\Delta_r = \rho(P_r) \ge 0$, by (4.14) and (3.18). Now $l(P) > p$, $l'(P) = p' = p + 2\widetilde{\varepsilon}$ and $l'(e_d) = l(e_d)$ for $d = i+1, \dots, j$ imply that

(4.19) $\qquad \Delta_r \in \{0, 1\}$ for $r = 1, 2$, $\qquad$ and $\Delta_r = 1$ for at most one r;

(4.20) $\qquad \widetilde{\varepsilon}$ is either $l(P) - p$ or $(l(P) - p)/2$.

Statement 4.11. *If $\widetilde{\varepsilon} = \varepsilon_2$ then AD' strictly includes AD.*

Proof. Consider P, i, j as above. By (4.19), we may assume that $\Delta_1 = 0$. Let $x := x_i$ and $e := e_i$ (if $i > 0$). We observe that

(4.21) $\qquad x$ is labeled, and if $i > 0$ then $\rho(x, e) = -1$.

This is trivial if $i = 0$, so assume that $i > 0$. By (4.14), $\Delta_1 = \rho(x_0, e_1) + \rho(x, e) = 1 + \rho(x, e)$, and $\Delta_1 = 0$ implies $\rho(x, e) = -1$. In particular, x is labeled.

Now consider two possible cases.

(i) $i < j$. Let $e' := e_{i+1}$. Since e' does not belong to a non-broken line, we know that $h(e') = 0$, $\gamma(e') = 0$ and e is unlabeled. If $x \in T$, e' is obviously labeled in Γ'. Let $x \notin T$, and let q be an arc in D entering x. Then $\tau = (x, e^q, e, \sigma^q, -1) \in \Pi$, by (4.21), whence $\tau \in \Pi'$ (by arguments as in Cases 1,4 of the proof of Statement 4.8). Also $\tau' = (x, e, e', 1, 1) \in \Pi'$ (as P is a T-line in Γ'). Now (3.1)(iii) for τ, τ' yields $(x, e^q, e', \sigma^q, 1) \in \Pi'$. Then $e \in AD'$.

(ii) $i = j$. Note that $x \notin T$ (otherwise $P = P_i$ for some $i \in \{1, 2\}$, whence $l(P) = p$). Let $e := e_i$ and $e' := e_{i+1}$, and let q be an arc in D entering x. Since x is labeled (by (4.21)), $\rho(x, e') \neq 0$, whence $\Delta_2 = \rho(x, e') + \rho(x_k, e_k)$ is even. In view of (4.19), $\Delta_2 = 0$. Then $\rho(x, e') = -1$. Now $(x, e^q, e, \sigma^q, -1) \in \Pi'$ and $(x, e, e', 1, 1) \in \Pi'$ imply $(x, e^q, e', \sigma^q, 1) \in \Pi'$, by (3.1)(iii). Hence, D' has the arc $q' = (x, x_{i+1})$ such that $e^{q'} = e'$ and $\sigma^{q'} = 1$. In view of $\rho(x, e') = -1$, the pair $(q', 1)$ is not in (D, σ). •

Thus, the dually increasing iteration enlarges the current digraph D, and Lemma 4.6 follows. • •

For purposes of Section 6 we need one more result.

Theorem 4.12. *For any integer $\widetilde{p}$, (2.1) has a half-integral optimal solution $\widetilde{\gamma}$.*

This is shown in [Ka2]. However, to make our description self-contained we give an alternative proof (under the assumption that a is positive).

It suffices to prove that, at the dually increasing iteration of the above algorithm, if the current p is an integer and γ is half-integral then $\widetilde{\varepsilon}$ as in (4.11) is a half-integer. First of all we observe that

(4.22) if $P = (x_0, e_1, x_1, \dots, e_k, x_k)$ is a T-path in Γ or a circuit in Γ then $\gamma(P)$ is an integer.

To see this, consider the potential function π on VG as in Section 2. Since γ is half-integral and a is integral, π is half-integral. By (2.6), for $i = 1, \dots, k$, $\gamma(e_i)$ is either $\pi(x_{i+1}) - \pi(x_i) - a(e_i)$ or $\pi(x_i) - \pi(x_{i+1}) - a(e_i)$ or $p - \pi(x_i) - \pi(x_{i+1}) - a(e_i)$. Thus, $2\gamma(e_i) \equiv 2\pi(x_{i+1}) - 2\pi(x_i) \pmod 2$ (taking into account that p is an integer). Then $2\gamma(P) \equiv \sum(\pi(x_{i+1}) - \pi(x_i) \,:\, i = 1, \dots, k) \equiv \pi(x_k) - \pi(x_0) \pmod 2$, whence $2\gamma(P)$ is even (as $\pi(x_0) = \pi(x_k) = 0$ if P is a T-path and $\pi(x_0) = \pi(x_k)$ if P is a circuit).

Suppose that $\widetilde{\varepsilon} = \varepsilon_1$. By (4.10), $\widetilde{\varepsilon} = -\gamma(e)/\rho(e)$ for some $e = xy \in E^-$ with $\rho(e) < 0$. If $\rho(e) = -1$, we are done. So assume that $\rho(e) = -2$. Then both x and y

are labeled. Let $Q = (y_0, u_1, y_1, \ldots, u_m, y_m)$ and $L = (z_0, w_1, z_1, \ldots, w_r, z_r)$ be paths in Γ such that $y_0, z_0 \in T$, $y_m = x$, $z_r = y$, and all edges in Q and L are labeled. Then $\gamma(u_i) = \gamma(w_j) = 0$. For the T-path or circuit P formed by Q, L and e, we have $\gamma(e) = \gamma(P) \in \mathbb{Z}$, by (4.22), whence $\widetilde{\varepsilon} \in \frac{1}{2}\mathbb{Z}$.

Now suppose that $\widetilde{\varepsilon} = \varepsilon_2$. Let $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$, i, j, P_1, P_2 be as in Statement 4.10. Apply (4.19)-(4.20). Let $\omega = l(P) - p$. If $\widetilde{\varepsilon} = \omega$, we are done. So assume that $\widetilde{\varepsilon} = \omega/2$. If $i = j$ then P lies entirely in Γ, and (4.22) implies that ω is an integer (whence $\widetilde{\varepsilon} \in \frac{1}{2}\mathbb{Z}$). So assume that $i < j$, $\Delta_1 = 0$ and $\Delta_2 \le 1$. Since $\widetilde{\varepsilon} = \omega/(2-\Delta_1-\Delta_2)$, $\Delta_2 = 0$. Hence (by (4.21)), there are paths $Q = (y_0, u_1, y_1, \ldots, u_m, y_m)$ and $L = (z_0, w_1, z_1, \ldots, w_r, z_r)$ in Γ such that $y_m, z_0 \in T$, $y_0 = x$, $z_r = y$, and all edges in Q and L are labeled. Let $R := P_1 \cdot Q^{-1}$ and $R' := L \cdot P_2$. Since $\gamma(u_d) = 0$ for all edges u_d in Q, (4.22) applied to R shows that $\gamma(P_1) \in \mathbb{Z}$. Similarly, $\gamma(P_2) \in \mathbb{Z}$. In addition, $\gamma(e_d) = 0$ for $d = i+1, \ldots, j$. Thus, $\gamma(P) \in \mathbb{Z}$, whence $\omega \in \mathbb{Z}$, and the result follows.

5. Capacity scaling algorithm

For $i \in \mathbb{Z}_+$ and $e \in EG$ define $c^i(e) := \lfloor c(e)/2^i \rfloor$, and let G^i be the subgraph of G with $VG^i = VG$ and $EG^i = \{e \in EG : c^i(e) > 0\}$. Thus, $G^0 = G$. Let I be the minimum $i \in \mathbb{Z}_+$ such that $c^i(e) = 0$ for all $e \in EG$.

It is convenient to assume that G is augmented by new edges $e_{ss'}$ connecting all distinct $s, s' \in T$; for such an $e = e_{ss'}$ we assign a large positive integral capacity $c(e) := 2^{I'}$ (with $I' \ge I$) and cost $a(e) := 1$. This leads to no loss of generality because one can see that any maximum multiflow for the new G, c has the property that $f(P) = c(e)$ for each $e = e_{ss'}$ and $P = (s, e, s')$.

The algorithm developed in this section consists of I stages. Let p be a large positive integer (so that any optimal solution to (1.3) is an optimal solution to (1.2) for G, T, c, a). As before, by an optimal solution to (1.3) we mean an appropriate regular function h. At the first stage, we find optimal solutions h^I and γ^I to (1.3) and its dual (2.1) for G^I, T, c^I, a, p. Due to the above assumption, EG^I consists just of the added edges $e = e_{ss'}$, and h^I and γ^I are determined trivially; namely, $h^I(e) = c(e)$ and $\gamma^I(e) = p - a(e) = p - 1$ for these e's.

In the input of the current, $(I - i + 1)$th, stage $(i < I)$, we are given optimal h^{i+1} and γ^{i+1} for $G^{i+1}, T, c^{i+1}, a, p$. Let $U^i := EG^i - EG^{i+1}$; and for $j = 0, \ldots, k = |U^i|$, let $c^i_j(e) := c^i(e) - 1$ $(= 2c^{i+1}(e))$ for $e = u_{j+1}, \ldots, u_k$, and $c^i_j(e) := c^i(e)$ for the other edges e in G^i. At the current, jth, iteration of the stage, we transform optimal solutions, $\widetilde{h}$ and $\widetilde{\gamma}$, to the problem with c^i_{j-1} into optimal solutions, $\widetilde{h}'$ and $\widetilde{\gamma}'$, to that with c^i_j (letting $\widetilde{h}$ and $\widetilde{\gamma}$ for $j = 1$ to be $2h^{i+1}$ and γ^{i+1}). Then $\widetilde{h}', \widetilde{\gamma}'$ for $j = k$ give optimal solutions for G^i, T, c^i, a, p.

Thus, the whole problem is reduced to the following auxiliary problem, which is to be solved, as a subroutine, at most $I|EG|$ times; for convenience we use the same notation as for the original problem.

(5.1) Given optimal h, γ for G, T, c, a, p (where $p \in \mathbb{Z}_+$ is large, c and a are positive and integral, γ is half-integral, and h is perfect with respect to T) and a fixed edge $u = z_1 z_2 \in EG$, find optimal $\widehat{h}, \widehat{\gamma}$ for $G, T, \widehat{c}, a, p$, where $\widehat{c}(u) = c(u) + 1$ and $\widehat{c}(e) = c(e)$ for all $e \in EG - \{u\}$.

(Here we assume, as above, that G contains the edges $e_{ss'}$.) Usually we will make $\widehat{h}$ also perfect. If $\gamma(u) = 0$, then h, γ are already optimal for the problem with $\widehat{c}$, so we assume that $\gamma(u) > 0$. Then u belongs to the l-graph $\Gamma = \Gamma^l$ for $l := a + \gamma$, and $h(u) = c(u)$, by (4.4). We form from Γ the following auxiliary graph G':

(5.2) (i) replace u by vertices t_1, t_2 and edges $u_i = t_i z_i$ of capacity $c(u) + 1$, $i = 1, 2$;

(ii) connect each two distinct $s, s' \in T$ by an edge $\overline{e} = \overline{e}_{ss'}$ of capacity $c(\overline{e}) = \infty$ (where $\overline{e}$ is different from $e_{ss'}$ above); let W be the set of these edges.

The capacity function for G' is denoted by c. We extend h to EG' by putting $h(u_i) := h(u)$ $(i = 1, 2)$, and $h(\overline{e}_{ss'}) := \sum(\lambda_j : P_j)$ connects s and s') for $\overline{e}_{ss'} \in W$, where $\mathcal{D} = \{(P_j, \lambda_j)\}$ is some fixed decomposition of h (in Γ) with all λ_j's half-integral.

Next, for $i = 1, 2$ define the attachment $s(z_i, u_i)$ to be $s(z_i, u)$. For $s \in T$ and $\overline{e} = \overline{e}_{ss'} \in W$ we introduce the special attachment $s(s, \overline{e}) := 0$ (extending $\langle T \rangle$ by the element 0); then the set $E(s)$ of edges in G' incident to s is partitioned into two subsets $E_s(s)$ and $E_0(s)$, and obviously $h(E_s(s)) = h(E_0(s))$. We consider $T'' := \{t_1, t_2\}$ to be the set of terminals for G' and, accordingly, consider the old terminals $s \in T$ as inner vertices. Then the function h on EG' is regular. Now we define the set Π of forks $(v, e, e', \sigma, \sigma')$ for $v \in VG' - T''$ by rule (4.5) (with G', T'' instead of G, T).

We formally extend a and γ to be zero on the set $W \cup \{u_1, u_2\}$, and define l on EG' to be $a + \gamma$. Next, since the initial function h on $E\Gamma$ is perfect and $h(u) = c(u)$,

(5.3) h is integral on u_1, u_2 and on each edge $e \in EG' - W$ with an end in T.

However, h need not be integral on W. We make H perfect for G', T'' by applying elimination procedures (C) or (D) from Section 3 to $\frac{1}{2}$-circuits in W (if any); this preserves the equality $h(E_s(s)) = h(E_0(s))$ for $s \in T$, so the resulting h is regular as well.

Let c', c'' on EG' be defined as in (4.6) with respect to the current capacities c; in particular, $c''(u_i) = c(u) + 1 = h(u_i) + 1$ $(i = 1, 2)$ and $c''(e) = \infty$ for $e \in W$.

The algorithm to solve (5.1) consists of at most two "primally increasing" iterations and of $O(|EG|)$ "dually increasing" ones. We attempt to find an augmenting path in each of two "networks" $N_1 = (G', T' = \{t_i\}, T'', h, \Pi, c', c'')$, $i = 1, 2$. Suppose that such a path exists for both N_1 and N_2. Two cases are possible.

Case 1. There is an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k)$ with $x_0 = t_1$ and $x_k = t_2$ (or with $x_0 = t_2$, $x_k = t_1$). By Corollary 3.3, we may assume that (P, Σ) satisfies (3.10). Make transformation of h as in (3.12).

Case 2. There is an augmenting path $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k)$ with $x_0 = x_k = t_1$ and an augmenting path $(Q = (y_0, w_1, y_1, \ldots, w_m, y_m), \Sigma' = (\sigma'_1, \ldots, \sigma'_m))$ with $y_0 = y_m = t_2$.

(a) P and Q are vertex-disjoint. We may assume that each of (P, Σ) and (Q, Σ') is as in (3.10). Transform h as in (3.12) along (P, Σ) and simultaneously along (Q, Σ').

(b) P and Q have a vertex $x_i = y_j =: v$ in common. Since $(v, e_i, e_{i+1}, \sigma_i, \sigma_{i+1}) \in \Pi$, at least one of $\tau = (x, e_i, w_j, \sigma_i, \sigma'_j)$ and $\tau' = (v, e_{i+1}, w_j, \sigma_{i+1}, \sigma'_j)$ is a fork, $\tau \in \Pi$ say. Hence, the concatenation of $(P, \Sigma)\langle x_0, x_i\rangle$ and the reverse to $(Q, \Sigma')\langle y_0, y_j\rangle$ is an augmenting path from t_1 and t_2. Then we proceed as in Case 1.

Let h' be the resulting function. Then either (i) $h'(u_1) = h'(u_2) = h(u) + 1$, or (ii) $h'(u_1) = h'(u_2) = h(u) + 1/2$. If (i) occurs, h', γ give the required solution $\widehat{h}, \widehat{\gamma}$ to (5.1) if we put $\widehat{h}(u) := h'(u_i)$. And if (ii) occurs, we apply to h', if needed, elimination procedures from Suction 3, which modify h' to be semi-perfect (with respect to $T' = \{t_1\}$ and T'' as above), and thereby both $h'(u_1)$ and $h'(u_2)$ are either preserved or increased by the same amount. If still $h'(u_i) < c(u) + 1$, we again attempt to find augmenting paths for both N_1 and N_2 (with h' instead of h), and if successful, we eventually obtain a function h'' with $h''(u_1) = h''(u_2) = c(u) + 1$, which induces a solution to (5.1). Now to complete the process we apply, if needed, elimination procedures that result in a perfect (with respect to G, T) solution $\widehat{h}$ to (5.1) with $\widehat{h}(u) = c(u) + 1 = \widehat{c}(u)$.

Now we suppose that, at the first or second of these iterations, for some N_j no augmenting path exists; without loss of generality let $j = 1$. Let (D, σ) and ρ be the corresponding signed digraph and function as in (3.14)-(3.15), respectively. (Then $q := (t_1, z_1) \in AD$ and $\sigma^q = 1$, while either u_2 is unlabeled or $q' := (z_2, t_2) \in AD$ and $\sigma^{q'} = -1$.) First of all, since G' contains the edges $e = e_{ss'}$ with a rather large $h(e)$, we may assume that

$$(5.4) \qquad h(\overline{e}) > 0 \quad \text{for all } \overline{e} \in W,$$

or, equivalently, $c'(\overline{e}) < h(\overline{e}) < c''(\overline{e})$ for all $\overline{e} \in W$. Hence, $(s, \overline{e}_{ss'}, \overline{e}_{ss''}, \sigma, \sigma') \in \Pi$ for any $s, s', s'' \in T$ and $\{\sigma, \sigma'\} = \{-1, 1\}$. This implies that if some vertex in T is labeled then all vertices in T and all edges in W are labeled. Hence, by (3.14)-(3.16),

$$\rho(\overline{e}) = 0 \quad \text{for all } \overline{e} \in W. \tag{5.5}$$

For $\varepsilon \in \mathbb{Q}_+$ let γ^ε be defined as in (4.8) with G' instead of Γ. Also we put

$$\gamma^\varepsilon(u) := \gamma(u) - \varepsilon \text{ if } u_2 \text{ is unlabeled, and } \gamma^\varepsilon(u) := \gamma(u) - 2\varepsilon \text{ otherwise.} \tag{5.6}$$

Let $l^\varepsilon := a + \gamma^\varepsilon$. Consider a T-line $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ in Γ, and form the circuit C by adding to P the edge $\overline{e} = \overline{e}_{ss'} \in W$ with $s = x_k$ and $s' = x_0$.

(a) Let $u \notin P$. By the definition of the attachment for $(s, \overline{e})$, $(s, e_k, \overline{e}, 1, 1)$ is a fork. Similarly, $(s', e_0, \overline{e}, 1, 1) \in \Pi$. Hence, C is a $(+)$-circuit in G' (see Definition 3.5); from Statement 3.6(iii) we have $\rho(C) \ge 0$, whence $\gamma^\varepsilon(C) \ge \gamma(C)$ and $l^\varepsilon(C) \ge l(C)$. Moreover, if $h(e_k) > 0$ then $(s, e_k, \overline{e}, -1, -1) \in \Pi$, by (5.4); similarly, $(s', e_0, \overline{e}, -1, -1) \in \Pi$ if $h(e_0) > 0$. Hence, if P is a strong line in Γ then C is a $(\pm)$-circuit in G', whence $\rho(C) = 0$ (by Statement 3.6(iii)) and $l^\varepsilon = l(C)$. Now, in view of (5.5), we have

$$\begin{aligned} &l^\varepsilon(P) \ge l(P) = p \text{ if } P \text{ is a } T\text{-line in } \Gamma; \\ &l^\varepsilon(P) = l(P) = p \text{ if } P \text{ is a strong line.} \end{aligned} \tag{5.7}$$

(b) Let $u \in P$. We may assume that $x_{i-1} = z_2$ and $x_i = z_1$ for some i (where z_1, z_2 are the ends of u). Form the path $P' = (t_1, u_1, x_i, e_{i+1}, \ldots, x_k, \overline{e}, x_0, e_1, \ldots, x_{i-1}, u_2, t_2)$ in G'. By arguments as above, P' is a $(+)$-T'-T''–line (whence $\rho(P) \ge 1$, by Statement 3.6(i)), and if P is a strong line then P' is a $(\pm)$-T'-T''–line (whence $\rho(P) = 1$, by Statement 3.6(ii)). Furthermore, $\rho(u_1) = 0$ (by (3.16)(ii)); $\rho(u_2) = 0$ if z_2 is unlabeled (since $\rho(z_2, u_2) = \rho(t_2, u_2) = 0$, by (3.14)); and $\rho(u_2) = -1$ if z_2 is labeled (since $\rho(t_2, u_2) = 0$, $\rho(z_2, u_2) \in \{1, -1\}$, and if $\rho(z_2, u_2) = 1$ then there would be an augmenting path from t_1 to t_2). Now (5.6) implies (5.7).

Easy arguments as in the proof of Statement 4.4 show that for a sufficiently small $\varepsilon \ge 0$, the transformation of γ to $\gamma' := \gamma^\varepsilon$ preserves p and all strong lines in Γ, whence h and γ' are optimal. Next, we define ε_1 as in (4.10) (letting $u \in E^-$, $\rho(u) := -1$ if u is unlabeled, and $\rho(u) = -2$ if u is labeled); and define $\widetilde{\varepsilon}$ as in (4.11), where ε_2 is the maximum $\varepsilon \le \varepsilon_1$ such that $l^{\varepsilon'}(P) \ge p$ for any T-path P in G and $0 \le \varepsilon' \le \varepsilon$.

Recall that p and a are integral. Assuming that γ is half-integral and arguing as in the proof of Theorem 4.12, one can show that $\widetilde{\varepsilon}$ is a half-integer, whence the new γ' is half-integral (in fact, we do not use this property in our algorithm). The only thing needed to be clarified is that $\gamma(u)/\rho(u)$ is half-integral. This is so if z_2 is unlabeled. And if z_2 is labeled, then there is a circuit $C=(x_0,e_1,x_1,\dots,e_k,x_k)$ with $e_1,\dots,e_k\in(EG'-\{u_1,u_2\})\cup\{u\}$ such that C contains u, and all other edges of C are labeled; let $u=e_1$, say. Then $\gamma(e_i)=0$ for $i=2,\dots,k$, all edges e_j of C are in $E\Gamma\cup W$, and $l(e_j)=a(e_j)+\gamma(e_j)$ is equal to some of $p-\pi(x_j)-\pi(x_{j+1})$ and $|\pi(x_j)-\pi(x_i)|$ (in particular, if $e_j\in W$ then $l(e_j)=\pi(x_j)=\pi(x_{j+1})=0$). This implies that $\gamma(u)$ is an integer.

After executing the "dually increasing" iteration we again try to find augmenting paths for both networks N_1' and N_2' (with G' induced by the new γ'), and so on. Repeating some arguments from Section 4, we can conclude that the new labeled digraph D' for N_1' strictly includes that for N_1; that N_2' has an augmenting path if N_2 does; and that the new labeled digraph for N_2' includes that for N_2 if N_2 has no augmenting path (an examination in details is left to the reader). This implies that the number of consecutive dually increasing iterations does not exceed $2|EG|$, and therefore the algorithm for solving (5.1) finishes in $O(|EG|)$ iterations.

Thus, the whole capacity scaling algorithm for (1.2) requires at most as many as $I=\log\parallel c\parallel_\infty$ times $O(|EG|^2)$ primally and dually increasing iterations.

6. Cost scaling algorithm

Assuming as before that c and a are integer-valued and positive, let I be the minimum $i\in\mathbb{Z}_+$ such that $2^i\geq a(e)$ for all $e\in EG$. For $e\in EG$ and $i\in\mathbb{Z}_+$ define $a^i(e):=\lceil a(e)/2^i\rceil$. Then $a^0(e)=a(e)$, $a^I(e)=1$, $a^i(e)>0$ and $2a^{i+1}(e)-a^i(e)$ is 0 or 1 for each $e\in EG$. Similarly to the previous section, without loss of generality we may assume that each two distinct $s,s'\in T$ are connected in G by an edge $e=e_{ss'}$ of large capacity and cost $a(e)=2^I$.

Fix a large positive p. The algorithm developed here consists of I stages. At the first stage we solve (1.3) for G,T,c,a^I,p. At the $(I-i+1)$th stage $(i<I)$ we find optimal h^i,γ^i for G,T,c,a^i,p, using optimal h^{i+1},γ^{i+1} for G,T,c,a^{i+1},p that were found at the previous stage.

We start by describing how to solve the first problem with a^I. We use a modification of the algorithm in Section 4. Let $a:=a^I$. In our case, $a(e)=1$ for all $e\in EG$. However, it is convenient to describe the modification for an arbitrary (positive integral) a. Let h,γ be optimal functions for the current $p:=p^l$, where $l:=a+\gamma$. We know that if the "primally increasing" iteration occurs for these h,γ in the algorithm of Section 4 then the value v_h increases by 1/2 or 1; thus, lengthy sequences of primally increasing iterations are possible. In fact, the modification combines such a sequence into one iteration that is executed in strongly polynomial time. As a result, we shall see that the total number of iterations turns out to be $O(a(EG))$. Thus, in case $a \equiv 1$ we get a strongly polynomial algorithm. The modification is close to the algorithm in [Ka1] which is based on the idea of constructing the so-called double covering digraphs over Γ.

More precisely, for the current l-graph $\Gamma = \Gamma^l$ we design the digraph $\mathcal{G} = (Z, A)$ as follows. As before, $E(v)$ ($E_s(v)$) denotes the set of edges in Γ incident to $v \in V\Gamma$ (resp. the set of edges $e \in E(v)$ with $s(v,e) = s$ for $s \in \langle T \rangle$). Let $\Omega(v)$ denote the set of $s \in \langle T \rangle$ such that $E_s(v) \neq \emptyset$. Then:

(6.1) (i) each $v \in V\Gamma$ produces $2|\Omega(v)|$ vertices v_s^1, v_s^2 ($s \in \Omega(v)$);

(ii) each $v \in V\Gamma - T$ produces $|\Omega(v)|(|\Omega(v)|-1)$ arcs $(v_s^2, v_{s'}^1)$ ($s, s' \in \Omega(v)$, $s \neq s'$), each of capacity ∞;

(iii) each edge $e = xy \in E\Gamma$ produces two arcs $(x_s^1, y_{s'}^2)$ and $(y_{s'}^1, x_s^2)$, each of capacity $c(e)$, where $e \in E_s(x)$ and $e \in E_{s'}(y)$.

The resulting digraph is just $\mathcal{G}$; we shall use notation c for the capacity function on A. Note that $\Omega(v) = \{v\}$ for each terminal $v \in T$; therefore, v produces two vertices in $\mathcal{G}$; we may denote v_v^1 and v_v^2 by v^1 and v^2 (the *first* and *second* copies of v in $\mathcal{G}$). We consider $T^1 := \{v^1 : v \in T\}$ ($T^2 := \{v^2 : v \in T\}$) to be the set of *sources* (*sinks*) of $\mathcal{G}$. For $v \in V\Gamma - T$ let $A_\infty(v)$ be the set of arcs as in (6.1)(ii), and let $A_\infty := \cup(A_\infty(v) : v \in V\Gamma - T)$. Denote by ϕ the natural mapping of $Z \cup (A - A_\infty)$ to $V\Gamma \cup E\Gamma$.

Next, a directed path in $\mathcal{G}$ from T^1 to T^2 is called a $T^1 - T^2$ *path*. The following property is important; this can be seen from the construction of $\mathcal{G}$ (cf. [Ka1]):

(6.2) (i) for a $T^1 - T^2$ path $P = (x_0, q_1, x_1, \ldots, q_k, x_k)$ in $\mathcal{G}$, the sequence $\phi(P) := (\phi(x_0), \phi(q_1), \phi(x_2), \phi(q_3), \ldots, \phi(x_{k-1}), \phi(q_k), \phi(x_k))$ is a T-line in Γ; in particular, $\phi(x_0) \neq \phi(x_k)$;

(ii) ϕ gives a one-to-one correspondence between the set of $T^1 - T^2$ paths in $\mathcal{G}$ and the set of T-lines in Γ.

Such a correspondence is extended, in a sense, to $T^1 - T^2$ flows for $(\mathcal{G}, c)$ and regular functions in Γ. Here by a $T^1 - T^2$ *flow* we mean a function $g : A \to \mathbb{Q}_+$ satisfying the capacity constraint $g(q) < c(q)$ for all $q \in A$ and the conservation condition $g^+(x) = g^-(x)$ for all $v \in Z - (T^1 \cup T^2)$, where

$$(6.3) \qquad g^+(x) := \sum(g(q) : q \in A^+(x)) \quad \text{and} \quad g^- := \sum(g(q) : q \in A^-(x));$$

here $A^+(x)$ ($A^-(x)$) is the set of arcs entering (leaving) x. We say that the sum of numbers $g^-(x)-g^+(x)=g^-(x)$ over all sources $v\in T^1$ is the value v_g of g. We observe the following:

(6.4) given a T^1-T^2 flow g in $\mathcal{G}$, define the function $h=h_g$ on $E\Gamma$ by $h(e):=\frac{1}{2}(g(q)+g(q'))$, where $e\in E\Gamma$ and $\{q,q'\}=\phi^{-1}(e)$; then h is c-admissible and regular, and $v_h=\frac{1}{2}v_g$.

Indeed, $h(e)=\frac{1}{2}(g(q)+g(q'))\le\frac{1}{2}(c(q)+c(q'))=c(e)$, therefore h is c-admissible. Consider $v\in V\Gamma-T$ and $s\in\Omega(v)$. By (6.1)(ii), $A_\infty(v)$ does not contain the arc going from v_s^2 to v_s^1; this implies that $g^+(v_s^2)\le\sum(g^-(v_{s'}^1)\ :\ s'\in\Omega(v)-\{s\})$ and $g^-(v_s^1)\le\sum(g^+(v_{s'}^2)\ :\ s'\in\Omega(v)-\{s\})$. Now

$$
\begin{aligned}
(6.5)\qquad & h(E_s(v))=\frac{1}{2}(g^+(v_s^2)+g^-(v_s^1))\ \text{ and}\\
& h(E(v)-E_s(v))=\frac{1}{2}\sum(g^+(v_{s'}^2)+g^-(v_{s'}^1)\ :\ s'\in\Omega(v)-\{s\})
\end{aligned}
$$

show that $h(E_s(v))\le h(E(v)-E_s(v))$, i.e. h is regular. The equality $v_h=\frac{1}{2}v_g$ is obvious. Note also that if g is integral then h_g is half-integral.

Conversely, given a regular function h on $E\Gamma$ and a decomposition $\mathcal{D}=\{(P_i,\lambda_i)\ :\ i=1,\dots,m\}$ of h (see Statement 2.3), we define the function $g=g_{h,\mathcal{D}}$ on A by

$$
(6.6)\qquad g(q):=\sum(\lambda_i\ :\ q\ \text{ is in }\ \phi^{-1}(P_i)\ \text{ or }\ \phi^{-1}(P_i^{-1}))\qquad\text{for }\ q\in A
$$

(this definition is correct, in view of (6.2)). It is easy to see that q is a (c-admissible) T^1-T^2 flow in $\mathcal{G}$, and $v_g=2v_h$. Moreover, for $g=g_{h,\mathcal{D}}$, h_g coincides with h.

Next, there is a relationship between active (augmenting) paths in Γ and in $\mathcal{G}$. Given γ, let A_0 be the set of arcs $q\in A$ such that either $q\in A_\infty$, or $q\in A-A_\infty$ and $\gamma(\phi(q))=0$. A path $P=(x_0,q_1,x_1,\dots,q_k,x_k)$ (not necessarily directed) in $\mathcal{G}$ is called *active* with respect to a T^1-T^2 flow g if: (i) $x_0\in T^1$, (ii) $q_1,\dots,q_k\in A_0$, and (iii) for $i=1,\dots,k$, $q_i=(x_{i-1},x_i)$ implies $g(q_i)<c(q_i)$, and $q_i=(x_i,x_{i-1})$ implies $g(q_i)>0$. Such a P is called *augmenting* if, in addition, $x_k\in T^2$. In what follows we deal with γ-*feasible* regular functions h on $E\Gamma$ and γ-*feasible* T^1-T^2 flows g in $\mathcal{G}$. This means that $h(e)=c(e)$ whenever $\gamma(e)>0$, and $g(q)=c(q)$ whenever $q\in A-A_0$. It is easy to see that the relations between T^1-T^2 flows and regular functions given in (6.4) and (6.6) preserve the property of being γ-feasible.

Statement 6.1. *Let g be a $T^1 - T^2$ flow in $\mathcal{G}$, and let $h = h_g$. The following are equivalent:*

(i) *there exists an augmenting path for $\mathcal{G}$ and g;*

(ii) *there exists an augmenting path for Γ and h.*

Proof. Suppose that there is an augmenting path $(P = (x_0, e_1, x_1, \dots, e_k, x_k), \Sigma = (\sigma_1, \dots, \sigma_k))$ for Γ and h (P uses only edges e with $\gamma(e) = 0$). Then for a sufficiently small $\varepsilon > 0$ the ε-transformation of h along P, defined by $h'(e) := \varepsilon \sum(\sigma_i : i \in \{1, \dots, k\}, e_i = e)$ for $e \in E\Gamma$, results in a regular, c-admissible, and γ-feasible function h with $v_{h'} = v_h + \varepsilon > v_h$. Choosing a decomposition $\mathcal{D}$ of h', form the $T^1 - T^2$ flow $g' = g_{h',\mathcal{D}}$ by rule (6.6). Then g' is γ-feasible, and $v_{g'} > v_g$. This means that g is not a maximum γ-feasible flow, and now a standard result in flow theory implies the existence of an augmenting path for g.

Now suppose that $P = (x_0, q_1, x_1, \dots, q_k, x_k)$ is a simple augmenting path for $\mathcal{G}, g$. Let $q_{j(1)}, \dots, q_{j(m)}$ be the arcs contained in $A - A_\infty$, and $j(1) < \dots < j(m)$. Form the path $Q = (y_0, u_1, y_1, \dots, u_m, y_m)$ in Γ, where $y_0 = \phi(x_0)$, $y_i = \phi(x_{j(i)})$ and $e_i = \phi(q_{j(i)})$, $i = 1, \dots, m$. Form the sequence $\Sigma = (\sigma_1, \dots, \sigma_m)$, where $\sigma_i = 1$ if $q_{j(i)}$ goes from $x_{j(i)-1}$ to $x_{j(i)}$, and $\sigma_i = -1$ otherwise. We assert that (Q, Σ) is augmenting for Γ, h. First of all, since $x_0 \in T^1$ and no arc enters x_0, we have $q_1 = (x_0, x_1)$, whence $\sigma_1 = 1$. Similarly, $\sigma_m = 1$.

Consider some $1 < i \le m$. For brevity, denote the part of P from $x_{j(i-1)-1}$ to $x_{j(i)}$ by $P' = (z_0, w_1, \dots, w_r, z_r)$; then $w_1, w_r \in A_0 - A_\infty$ and $w_2, \dots, w_{r-1} \in A_\infty(v)$, where $v := \phi(z_1) = \dots = \phi(z_{r-1})$. Let $e := \phi(w_1)$, $e' := \phi(w_r)$, $\sigma := \sigma_{i-1}$ and $\sigma' := \sigma_i$. We have to show that $\tau = (v, e, e', \sigma, \sigma') \in \Pi$. Let $s := s(v, e)$ and $s' := s(v, e')$. Consider possible cases.

(i) $r = 2$. Then either $w_1 = (z_0, z_1)$ and $w_2 = (z_2, z_1)$, or $w_1 = (z_1, z_0)$ and $w_2 = (z_1, z_2)$. In both cases we have $s = s'$ (i.e., $e, e' \in E_s(v)$) and $\sigma = -\sigma'$. Then $\tau \in \Pi$.

(ii) $r > 2$, $z_1 = v_s^2$ and $z_{r-1} = v_{s'}^1$. Then $w_1 = (z_0, z_1)$ and $w_2 = (z_{r-1}, z_r)$, whence $\sigma = \sigma' = 1$. If $s \ne s'$ then $\tau \in \Pi$ is obvious, so assume that $s = s'$. Since $(v_s^2, v_s^1) \notin A$, there is $1 < d < r$ such that w_d goes from $v_t^2 = z_d$ to $v_{t'}^1 = z_{d-1}$ for some distinct $t, t' \in \Omega(v) - \{s\}$; then $g(w_d) > 0$ (as P is active). Now we can deduce that s is not tight for v, h (for otherwise (6.5) and the equality $h(E_s(v)) = h(E(v) - E_s(v))$ would easily imply that $g(v_t^2, v_{t'}^1) = 0$ for all $t, t' \in \Omega(v) - \{s\}$). Thus, $\tau \in \Pi$.

We leave it to the reader to examine the remaining cases in which $r > 2$ and either $z_1 = v_s^1$ and $z_{r-1} = v_{s'}^2$, or $z_1 = v_s^1$ and $z_{r-1} = v_{s'}^1$, or $z_1 = v_s^2$ and $z_{r-1} = v_{s'}^2$, using arguments similar to those in case (ii). •

Now we are able to describe the above-mentioned modified algorithm. Given h and γ, design $\mathcal{G}$ over the current Γ. Using a strongly polynomial maximum flow subroutine, we find a $T^1 - T^2$ flow g' in $\mathcal{G}$ that has the maximum value $v_{g'}$, provided that g' is γ-feasible, i.e., $g'(q) = c(q)$ for all $q \in A - A_0$; such a flow exists because $g = g_{h,\mathcal{D}}$ is γ-feasible, where $\mathcal{D}$ is a decomposition of h. (For a survey of flow algorithms, see, e.g., [GTT].) Moreover, since all capacities in $\mathcal{G}$ are integral or infinite, we can find an integral g'. Then $h' := h_{g'}$ is half-integral, by (6.4), and h', γ are optimal for the current p. This h' is the resulting function of the current iteration. By Statement 6.1, no augmenting path for Γ, h', γ exists (since there is no augmenting path for $\mathcal{G}, g'$). So the next iteration is dually increasing; it consists of transforming γ as in the algorithm of Section 4, and yields a half-integral γ' such that $p^{l'} \geq p + 1$, where $l' := a + \gamma'$.

Let $p_0 < p_1 < \ldots < p_M$ be the sequence of values of the parameter p on the iterations (assuming that the last iteration finishes with $\widetilde{\varepsilon} = \infty$). We observe that the number of iterations is at most $2M$ (as $p_{i+1} \geq p_i + 1$, and there are no two consecutive primally increasing iterations). To estimate M, consider the last primally increasing iteration; let $p_{M'}$ be the value of p at this iteration. Then $M - M'$ is $O(|EG|)$ (in view of Lemma 4.6 and the fact that all iterations after that we consider are dually increasing). Furthermore, $p_{M'} \geq M'$. We show that $p := p_{M'}$ is at most $2a(EG)$, whence the number of iterations of the modified algorithm is $O(a(EG))$.

Indeed, let $(P = (x_0, e_1, x_1, \ldots, e_k, x_k), \Sigma = (\sigma_1, \ldots, \sigma_k))$ be an augmenting path in Γ at the above-mentioned iteration. By (3.9)-(3.10), we may assume that each edge of Γ occurs in P at most twice. Furthermore, $l(P) = a(P)$ since $\gamma(e_i) = 0$ for $i = 1, \ldots, k$. If $x_0 \neq x_k$ then $l(P) \geq p$, whence $p \leq a(P) \leq 2a(EG)$. Now let $x_0 = x_k =: s$. We assert that P must have a vertex x_i not in V_s (V_s was defined in Section 2). For a contradiction, suppose that $x_0, \ldots, x_k \in V_s$. Consider the maximum j such that either (i) $\sigma_j = 1$ and $\pi(x_j) > \pi(x_{j-1})$, or (ii) $\sigma_j = -1$ and $\pi(x_j) < \pi(x_{j-1})$ (such a j exists since $\sigma_1 = 1$ and $\pi(x_1) > \pi(x_0) = 0$). Since $\sigma_k = 1$ and $0 = \pi(x_k) < \pi(x_{k-1})$, we have $j < k$. Let $e := e_j$, $e' := e_{j+1}$, $\sigma := \sigma_j$ and $\sigma' := \sigma_{j+1}$. Note that both s and $-s$ are tight for $v := x_j$, and that $\tau = (v, e, e', \sigma, \sigma') \in \Pi$. In case (i), we have $e \in E_s(v)$ (as $\pi(x_j) > \pi(x_{j-1})$); hence, $e' \in E_s(v)$ would imply $\sigma' = -1$ and $\pi(x_{j+1}) < \pi(x_j)$, while $e' \in E_{-s}(v)$ would imply $\sigma' = 1$ and $\pi(x_{j+1}) > \pi(x_j)$. This contradicts to the maximality of j. Similarly, j cannot be maximum in case (ii). Thus, some x_i is not in V_s. Then $l(P) \geq 2\text{dist}_l(s, x_i) \geq p$, whence $p \leq a(P) \leq 2a(EG)$, as required.

Returning to the cost scaling algorithm, we conclude that the first stage requires $O(a^I(EG))$, or $O(|EG|)$, iterations of the above modified algorithm, and therefore it is solvable in strongly polynomial time.

Now we describe the general, $(I - i + 1)$th, stage of the algorithm, $i < I$. Let $U^i := \{e \in EG : 2a^{i+1}(e) > a^i(e)\}$; and for $j = 1, \ldots, k = |U^i|$, let $a^i_j(e) := a^i(e)$ for

$e = u_1, \ldots, u_j$, and $a^i_j(e) := 2a^{i+1}(e)$ for the other edges e in G. At the current, jth, iteration of the stage, we transform optimal solutions, $\widetilde{h}$ and $\widetilde{\gamma}$, to the problem with a^i_{j-1} into optimal solutions, $\widetilde{h}'$ and $\widetilde{\gamma}'$, to the problem with a^i_j (letting $\widetilde{h}$ and $\widetilde{\gamma}$ for $j = 1$ to be h^{i+1} and $2\gamma^{i+1}$). Then $\widetilde{h}'$ and $\widetilde{\gamma}'$ for $j = k$ give optimal solutions for a^i.

Thus, the stage is reduced to the following auxiliary problem, to be solved, as a subroutine, at most $I|EG|$ times; for convenience we use the same notations as for the original problem.

(6.7) Given optimal h, γ for G, T, c, a, p (where $p \in \mathbb{Z}_+$ is large, c and a are positive and integral, γ is half-integral, and h is perfect with respect to T) and a fixed edge $u = z_1 z_2 \in EG$ with $a(u) > 1$, find optimal $\widehat{h}, \widehat{\gamma}$ for $G, T, c, \widehat{a}, p$, where $\widehat{a}(u) = a(u) - 1$ and $\widehat{a}(e) = a(e)$ for all $e \in EG - \{u\}$.

(Cf. (5.1)). As above, we assume that G contains the edges $e_{ss'}$. Usually we will make $\widehat{h}$ perfect. Put $a(u) := a(u) - 1$; this decreases $l(u)$ by 1. If the new l satisfies the inequality $\mathrm{dist}_l(s, s') \geq p$ for all distinct $s, s' \in T$ (which can happen when u is not in $\Gamma = \Gamma^l$ for the previous $l := a + \gamma$) then h, γ are optimal for the problem with $\widehat{a}$. If not, we add to $\gamma(u)$ the minimum number (1 or 1/2) that restores the l-distance p for such s, s'. Note that if $h(u) = c(u)$ then h and the new γ are optimal for the problem in question. So we may assume that u is in $\Gamma = \Gamma^l$, $h(u) < c(u)$ and $\gamma(u) \in \{1, 1/2\}$ (since $h(u) < c(u)$ implies that $\gamma(u) = 0$ for the old γ). Thus, the complementary slackness condition in (4.4) is violated for u and holds for the other edges.

We form the auxiliary graph G', extend the corresponding functions and attachments as it is done in Section 5, and consider $T'' := \{t_1, t_2\}$ to be the set of terminals in G'. Next, we form the digraph $\mathcal{G} = (Z, A)$ for G', T'' instead of Γ, T. As a result, $\mathcal{G}$ has the set $T^1 = \{b, d\}$ of sources and the set $T^2 = \{b', d'\}$ of sinks, where b (b') is the first (second) copy of t_1 and d (d') is the first (second) copy of t_2. We form a $T^1 - T^2$ flow g in $\mathcal{G}$ from h and its decomposition $\mathcal{D}$. Let q (q') be the arc derived from u_1 and leaving b (entering b'), and w (w') be the arc derived from u_2 and leaving d (entering d'). Then

$$g(q) = g(q') = g(w) = g(w') = h(u). \tag{6.8}$$

Our goal is to find a γ-feasible integral $T^1 - T^2$ flow g' such that

$$g'(q) + g'(q') = g'(w) + g'(w'), \text{ and} \tag{6.9}$$

(6.10) for g', at least one of the following is true:

(i) there is no augmenting path from b to $\{b', d'\}$,

(ii) there is no augmenting path from d to $\{b', d'\}$.

Then (6.9) implies that $h'(u_1) = h'(u_2)$ for $h' = h_{g'}$ (in view of (6.4)), and (6.10) ensures that for some $i \in \{1,2\}$ there is no augmenting path from $T' = \{t_i\}$ to $T'' = \{t_1, t_2\}$ with respect to h' (by statement similar to Statement 6.1).

Finding g' is reduced to at most two maximum flow problems, as follows.

(a) Add to $\mathcal{G}$ an arc $\widehat{e}$ from b' to d with the lower capacity $h(u)$ and the upper capacity ∞, and for the resulting network find a maximum γ-feasible flow g_1 from b to d'. Then:

(6.11) $$g_1(q') = g_1(w) = g_1(\widehat{e}) \geq h(u); \quad \text{and}$$

(6.12) there is $X \subset Z$ such that: $b \in X \not\ni d'$; $g_1(e) = c(e)$ for any $e \in (X, \overline{X})$; $g_1(e) = 0$ for any $e \in (\overline{X}, X) \cap A_0$; $g_1(e) = c(e)$ for any $e \in (\overline{X}, X) - A_0$; and $b' \in X \not\ni d$ is impossible;

here $(Y, \overline{Y})$ denotes the set of arcs in $\mathcal{G}$ (without $\widehat{e}$) going from $Y \subset Z$ to $\overline{Y} := Z - Y$. (The latter property in (6.12) is provided by the assignment of the infinite upper capacity for $\widehat{e}$.) If $d \in X \not\ni b'$ then, obviously, g_1 is just the required g' satisfying (6.9)-(6.10). So assume that either $d, b' \in X$ or $d, b' \notin X$.

(b) Add to $\mathcal{G}$ an arc $\widehat{e}'$ from d' to b with the lower capacity $g_1(q)$ ($= g_1(w')$) and the upper capacity ∞, and for the resulting network find a maximum γ-feasible flow g_2 from d to b'. Then there is $Y \subset Z$ with $d \in Y \not\ni b'$ having the properties similar to those for X in (6.12). Note that $g_2(q) = g_2(w') \geq g_1(q) = g_1(w')$ (by the assignment of the lower capacity for $\widehat{e}'$). This together with (6.12) and the fact that $|\{d, b'\} \cap X| \neq 1$ easily implies that $g_2(e) = g_1(e)$ holds for any $e \in (X, \overline{X}) \cup (\overline{X}, X)$. Put $g' := g_2$. Then (6.9) is obvious, and (6.10) follows from the above properties of the cuts in $\mathcal{G}$ induced by X and by Y.

Thus, this "primally increasing" iteration finishes with optimal h', γ such that either (i) $h'(u_1) = h'(u_2) = c(u)$, or (ii) $h'(u_1) = h'(u_2) < c(u)$ and for some $i \in \{1,2\}$ there is no augmenting path from $T' := \{t_i\}$ to $T'' := \{t_1, t_2\}$. In case (i), we have a solution to (6.7). In case (ii), we execute the "dually increasing" iteration with respect to T', in a way similar to that described in Section 5; then repeat the process. Now the fact that $\gamma(u) \leq 1$ shows that at most two dually increasing iterations can happen.

Thus, (6.7) is solvable in strongly polynomial time. This and the strong polynomiality for the first stage imply that the above cost scaling algorithm runs in time $I = \| a \|_\infty$ times a polynomial in $|EG|$.

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